Conformal Defects in Nonlocal $O(N)$-invariant Theories

Leonardo S. Cardinale (M1 intern)*[1] and Lorenzo Bianchi (supervisor)†‡[2]

[1] *Département de Physique, Ecole Normale Supérieure – PSL*
[2] *Dipartimento di Fisica, Università di Torino and INFN – Sezione di Torino*

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A modern approach to studying quantum field theories (QFTs) involves constraining renormalization group (RG) flows between ultraviolet and infrared fixed points, which are typically conformal field theories (CFTs). In this work, we study nonlocal $O(N)$-invariant CFTs, which are of interest to high energy physics and yield applications in condensed matter physics. A special case is that of the long-range Ising fixed point (LRI), an extensively studied CFT whose action is given by a generalized free field (GFF) perturbed by a marginally relevant ϕ^4 interaction. Our study focuses on building non-trivial conformal defects in nonlocal $O(N)$-invariant theories, in particular the LRI and its vector counterpart referred to here as the long-range $O(N)$ model. It turns out that recent work on the localized magnetic field and the quadratic surface defect in the $O(N)$ model [1–5] can be generalized with ease to its nonlocal cousin, and we do so here; however, non-trivial RG flows are only observed close to dimension $d = 4$. In keeping with the picture of interacting Heisenberg spins in physical space, we suggest a new model which allows one to place the critical point close to $d = 3$. This involves a new GFF living on a defect interacting with the bulk GFF from the LRI. Tuning the fractional Laplacian parameter of the defect GFF allows one to trigger RG flows at arbitrary d, and the $d = 3$ case can be made to respect unitarity constraints. Incidentally, some of these theories are studied in a free bulk, and demonstrate that nonlocality provides a loophole to an important result in [6] according to which there can be no non-trivial defects in integer dimensions $d \leq 4$ for a free (local) theory.

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*leonardo.cardinale@ens.psl.eu
†lorenzo.bianchi@to.infn.it
‡I also extensively worked with Elia de Sabbata, Lorenzo Bianchi's PhD student, who is a main collaborator on a paper which we will write regarding the same subject.

1 Introduction

1.1 Lightning review of Quantum Field Theory

Before addressing the models studied in this work, we provide a very brief overview of the main tools from QFT which are useful to understanding most of the computations we've conducted. The usual way of defining a QFT is with an action S, traditionally given as an integral over a Lagrangian density:

$$S[\phi] = \int d^d x \mathcal{L}\left(\phi, \partial_\mu \phi\right) \tag{1.1}$$

where ϕ is a scalar field (we shall only be looking at Poincaré scalars in this work, and the generalization to $O(N)$ vector models is straightforward). While, classically, one varies the action with respect to the fields to derive equations of motion which are subsequently solved, quantum mechanics imposes a probabilistic paradigm which adds quantum fluctuations. In a Euclidean signature (making the change of variables $t \mapsto it$ from Minkowski space for example), the path integral formalism leads to the following expression for correlation functions of ϕ (see sections 6-8 in [7] for a derivation in Lorentzian signature, sections 1.2 and 1.3 in [8] for a derivation in Euclidean signature with a lattice twist):

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = \frac{1}{Z} \int \mathcal{D}\phi \phi(x_1) \dots \phi(x_n) e^{-S[\phi]}, \quad \mathcal{D}\phi = \prod_{x \in \text{ lattice}} d\phi(x) \tag{1.2}$$

where Z is a normalization factor, $\mathcal{D}\phi$ is a measure over the functional space ϕ lives in (which we won't specify here), and we've included one way to define the integration measure as a continuum limit on the lattice. It's worth nothing that this path integral is identical to what one would get in the canonical ensemble in statistical mechanics, and as such Z is essentially a partition function.

Correlation functions are the main observables of a QFT, from which one can typically extract scattering amplitudes and make experimental predictions (via the LSZ reduction formula, see section 5 in [7]). While the path integral (1.2) is in general incredibly hard to compute exactly, it turns out that there are specific cases where it becomes comparatively simple. The epitome of a computable path integral is that of a Gaussian – or free – theory, defined as one whose action is quadratic in the fields. Such an action for a single field can be written

$$S[\phi] = \frac{1}{2} \int d^d x d^d y \phi(x) D(x, y) \phi(y) \tag{1.3}$$

and the path integral subsequently becomes the infinite-dimensional generalization of well-known Gaussian integrals. We'll also refer to fields in the above action as generalized free fields (GFF for short). Let $G(x)$ be the solution of

$$\int D(x, y) G(y) dy = \delta(y) \tag{1.4}$$

Such a G is called a Green's function, or propagator in the context of QFT. An important result known as Wick's theorem tells us that for such a free theory [9]:

$$\langle \phi(x_1) \dots \phi(x_{2n}) \rangle = \sum_{\{i_n\}} G(x_{i_1} - x_{i_2}) G(x_{i_3} - x_{i_4}) \dots G(x_{i_{2n-1}} - x_{i_{2n}}) \tag{1.5}$$

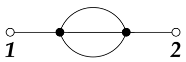
In other words, a free correlator is only non-zero for an even number of field insertions, and is given by the sum over all possible pairings of fields of products of propagators. Most theories' actions are written as the sum of a free, Gaussian term, and an interaction term, which is typically a polynomial in the fields. For example, one can take the famous example of ϕ^4 interactions, which is very relevant to this work:

$$S = S_0 + S_{int} = \frac{1}{2}\int d^dxd^dy\phi(x)D(x,y)\phi(y) + \frac{\lambda}{4!}\int d^dx\phi(x)^4 \tag{1.6}$$

The main method for calculating correlators of this interacting model is perturbation theory, taking λ to be "small". One then gets, adding a zero subscript to signal expectation values taken against the free theory [10]:

$$\begin{aligned}\langle\phi(x_1)\dots\phi(x_n)\rangle &= \frac{\langle\phi(x_1)\dots\phi(x_n)e^{-\frac{\lambda}{4!}\int d^dx\phi^4}\rangle_0}{\langle e^{-\frac{\lambda}{4!}\int d^dx\phi^4}\rangle_0} \\ &= \frac{\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^k}{4!^k k!}\int d^dy_1\dots d^dy_k\langle\phi(x_1)\dots\phi(x_n)\phi^4(y_1)\dots\phi^4(y_k)\rangle_0}{\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^k}{4!^k k!}\int d^dy_1\dots d^dy_k\langle\phi^4(y_1)\dots\phi^4(y_k)\rangle_0}\end{aligned} \tag{1.7}$$ [1]

Wick's theorem then applies to the free correlators in the integrands. Each term in the sums can be represented as a so-called *Feynman diagram*. In position space as above, a Feynman diagram is a graph where the edges between two vertices represent a propagator, external vertices represent the points at which we take the correlator of the fields, and internal vertices represent interaction terms we integrate over. Conversely, in momentum space external lines represent incoming and outgoing momenta, internal lines represent propagators, and vertices as before signal an interaction. For example in position space:



$$= \frac{\lambda^2}{4!}\int d^dy_1d^dy_2G(y_1-x_1)G(y_1-y_2)^3G(x_2-y_2) \tag{1.8}$$ [2]

It turns out that dividing by the partition function – the denominator in (1.7) – cancels disconnected diagrams (disconnected in the usual graph-theoretical sense). In other words, only connected diagrams contribute, and summing over more and more of those diagrams each multiplied by a combinatorial factor builds the full correlation function (a proof of this statement can be found in section 9 of [7]).

Often times, higher order diagrams involving loops – such as the one in equation (1.8) – introduce divergences to these correlators, which can sometimes be cured using the tools of *renormalization*, combined with the framework of the *renormalization group*. In order to renormalize a theory, it is important to understand its dependence on a mass scale μ. In ϕ^4 theory, this amounts to studying the mass dimension of the coupling λ, denoted $[\lambda]$, assuming natural units for which $\hbar = c = 1$. If $[\lambda] > 0$, the coupling is said to be relevant (i.e. relevant at low energies or large distances); if $[\lambda] < 0$, it is irrelevant (at low energies); otherwise, $[\lambda] = 0$ and it is said to be classically marginal (classically, the interaction is scale-invariant – quantum corrections may change this however). When looking at the theory with $[\lambda] = \varepsilon$ i.e. for a marginally relevant coupling, loop diagrams will typically contain poles in ε, of order at most the power of g in perturbation theory. Renormalization is typically conducted by adding counterterms to the action of a theory which will cancel out these poles (see [7] for a general discussion, and [9] for the renormalization of ϕ^4 theory).

Counterterms amount to redefining all of the quantities in the action as *bare* quantities, related to *renormalized* quantities via a Laurent series with poles in ε. For ϕ^4 theory, one thus defines:

$$\phi = Z_\phi(g,\varepsilon)[\phi], \quad m_0^2 = Z_{m^2}(g,\varepsilon)m^2, \quad \lambda_0 = Z_\lambda(g,\varepsilon)g\mu^\varepsilon \tag{1.9}$$ [3]

where we've included the possibility of a massive theory, the mass appearing in the function $D(x,y)$ in the free theory. Quantities appearing with a zero index are bare, while those without an index are renormalized; fields appearing with brackets are the renormalized fields, and their renormalization is referred to as *wavefunction renormalization*. These $Z(g,\varepsilon)$ functions can be written

$$Z(g,\varepsilon) = 1 + \sum_n \frac{a_n}{\varepsilon^n} \tag{1.10}$$

where the a_n are chosen to cancel the poles in divergent diagrams, such that the theory in the renormalized quantities remains finite. Typically, diagrams involving a Green's function evaluated at zero spacing $G(0)$ – so-called tadpole diagrams – involve mass renormalization, while those which blow up when internal vertices get closer and closer to one another require coupling or wavefunction renormalization.

[1] The permutation of summation and integral signs is highly non-trivial. In fact, in general the series is divergent. Nevertheless, this is viewed as an asymptotic series, whose first terms can yield useful predictions.

[2] All of the Feynman diagrams in this section are taken from [9, 10].

[3] Incidentally, one often finds $\sqrt{Z_\phi}$ instead of Z_ϕ in the literature, however the above convention fits best with the work conducted below and the conventions in relevant papers.

Renormalization might seem like a quaint mathematical trick. However, more intuition can be developed from the Wilsonian point of view of the renormalization group (see chapter 23 of [11] or the lecture notes [12] for a presentation). It consists in adopting the seemingly reasonable point of view according to which physics is scale-dependent. Indeed, while a fluid is nothing but a macroscopic collection of elementary particles, one need not describe it using the full might of QFT. However, the macroscopic 'effective theory' should be related to the high energy theory by some non-trivial transformation, which translates a kind of coarse-graining from the ultraviolet (UV) to the infrared (IR), or from high to low energies, these terms being standard in the literature. At the level of an action in QFT, this transformation amounts to modifying the values for the masses and couplings of the initial theory, which 'run' under a renormalization group flow. The *ad hoc* introduction of new masses and couplings can hence be interpreted as the consequence of a change of scale. Nevertheless, correlation functions in the bare theory should not depend on the mass scale μ introduced. This statement yields the Callan-Symanzik equation, which is central to the renormalization group:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta_\lambda(g)\frac{\partial}{\partial g} + n\gamma_\phi\right]\langle[\phi](x_1)\dots[\phi](x_n)\rangle = 0 \tag{1.11}$$

where we've introduced the β-function for the λ coupling and the so-called 'anomalous dimension' of ϕ, defined by:

$$\beta_\lambda(g) = \frac{\partial g}{\partial\log\mu}, \gamma_\phi = \frac{1}{Z_\phi}\frac{dZ_\phi}{d\log\mu} \tag{1.12}$$

The β-function expresses the scale-dependence of the running coupling g. Scale invariance is achieved for $g = g_*$ satisfying $\beta_\lambda(g_*) = 0$, and the theory at g_* is said to be a fixed point for the RG flow. Now consider the two-point function $F(x, g, \mu) = \langle[\phi](0)[\phi](x)\rangle$. Let's introduce a dimensionless parameter s such that a coordinate x satisfies $|x| = s/\mu$. For dimensional reasons, $F = f(s, g)|x|^{-2\Delta_\phi}$ where Δ_ϕ is the mass dimension of ϕ (reasoning and notation from [13]). At the fixed point, the Callan-Symanzik equation for F yields:

$$s\frac{\partial f}{\partial s} = -2\gamma_\phi f \tag{1.13}$$

This entails the following behavior at the fixed point

$$\langle\phi(x)\phi(0)\rangle \sim \frac{1}{|x|^{2(\Delta_\phi+\gamma_\phi)}} \tag{1.14}$$

This justifies why we've named γ_ϕ the anomalous dimension of ϕ. It is the anomalous contribution to the mass dimension of ϕ gained at the fixed point. It turns out that the scaling in equation (1.14) is typical of a class of theories called conformal field theories (CFTs for short).

1.2 A word on Conformal Field Theory and defects

Quantum field theory can be viewed as a Lorentz-invariant apparatus used to compute correlation functions. It involves quantum fields which transform under infinite-dimensional unitary representations of the Poincaré group (translations $\rtimes$ Lorentz group) obeying certain causality constraints (see [14] for details). A natural question is therefore whether one can extend the Poincaré group in a non-trivial fashion, and build a field theory with the resulting group. It turns out that the ways to do this are surprisingly limited (see section 2 in [15] for example), and essentially amount to the construction of the conformal group.

A conformal transformation $x \mapsto x'$ is loosely defined as a transformation which preserves angles. Formally, it is defined as a transformation which maps the metric tensor $g_{\mu\nu}$ to $\Omega(x)g_{\mu\nu}$ for a certain scale factor $\Omega(x)$. The set of these transformations forms a group under composition, of which the Poincaré group ($\Omega = 1$) is a subgroup. In a Euclidean metric δ_{mn} and dimensions strictly greater than 2, as is the case in this work, the conformal group is made up of four classes of transformations: translations, rotations, dilatations and special conformal transformations. The first two are just Poincaré transformations, and conformal symmetry adds dilatations $x \mapsto \lambda x$ and special conformal transformations (see [15] for expression). Special conformal transformations, while non-trivial to illustrate, can be understood as transformations which tend to map planes to spheres (the stereographic projection is an example of this).

By studying the Lie algebra associated with the conformal group, one can determine the conformal algebra to be $\mathfrak{so}(d+2)$; incidentally, this is why the Euclidean conformal group is often colloquially referred to as $SO(d+2)$. Among the relevant commutation relations, two are of particular note [16]:

$$\begin{cases}[D, P_m] = P_m \\ [D, K_m] = -K_m\end{cases} \tag{1.15}$$

where P_m and K_m are the translation and special conformal generators respectively. This set of commutation relations suggests that translation and special conformal generators act as ladder operators in the eigenspace of D. We'll typically be looking at operators which have a specific scaling dimension Δ – the eigenvalue associated with D, coinciding with a mass dimension – and as such shall be working in these eigenspaces. Operators annihilated by K_m are be called *primary* operators, while those obtained by acting on a primary with P_m are referred to as *descendant* operators.

It turns out that conformal symmetry heavily constrains the shape correlation functions of primaries can take, and this is in fact one of the main advantages of working in a CFT. Consider three local primary operators $\mathcal{O}_1(x_1), \mathcal{O}_2(x_2)$ and $\mathcal{O}_3(x_3)$ without spin. Conformal symmetry requires in particular that (look at section 4.3.1 from the classic reference [17] for a justification):

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \begin{cases} \frac{C}{|x_1-x_2|^{2\Delta_1}} \text{ if } \Delta_1 = \Delta_2 \\ 0 \text{ otherwise} \end{cases} \tag{1.16}$$

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{\lambda_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}} \tag{1.17}$$

Let's now consider the case of a defect. A defect consists in a p-dimensional subspace of a given d-dimensional bulk, on which one modifies the action, typically with a term that, taken in isolation, is conformal. This amounts to adding to the bulk action S_{bulk} an action S_{defect} which is a p-dimensional integral. The main defects we'll consider here will be of the two following types:

$$S_{\text{defect}} = h_0 \int_p d^p x \phi(x)^k, \quad S_{\text{defect}} = \frac{1}{2}\int_p d^p x d^p y \psi(x) D(x,y) \psi(y) + \frac{g_0}{2}\int d^p x \phi(x)^a \psi(x)^b \tag{1.18}$$

where for $k=1$, the first one is referred to as a localized magnetic field when $p=1$, by analogy with the usual coupling of Ising spins to an external magnetic field h_0. It's worth noting that we'll also occasionally look at circular defects, which are useful in that they do not present tadpole divergences which tend to appear in the computation of a partition function.

One might ask why defects are worth studying. Some examples where defects are relevant include: the study of phase transitions, such as in Yang-Mills theories via Wilson loops (including QCD and the elusive color confinement, see [8]), the understanding of conserved charges obtained by integrating a symmetry current over some sub-region of space viewed as a defect. More relevant to experimental setups, systems in condensed matter can involve defects, typically variations on magnetic impurities [2]. However, defects might also seem to be of intrinsic interest to field theorists, since they offer a generalization, and perhaps an alternative to local operators which appear in traditional QFT [5].

It turns out that adding a defect breaks certain symmetries. Indeed, upon introducing a defect, we no longer have translation symmetry along the direction orthogonal to the defect, nor do we have rotation symmetry for rotations mixing the coordinates parallel and orthogonal to it. In other words, the symmetry breaking can be written $SO(d+2) \to SO(d-p) \times SO(p+2)$. This essentially means that the defect theory is a CFT, and that in the bulk we maintain rotation symmetry along directions orthogonal to the defect. Hence, the results for the two-point and three-point functions derived previously are no longer valid. Nevertheless, relatively simple statements can still be made about correlators in some situations.

In a theory with a defect, local operators $\mathcal{O}(x)$ are classified as either bulk or defect operators (we'll add a hat over defect operators). Since the theory on the defect is a CFT, all of the results regarding CFT correlators hold when studying operators on the defect. However, the story is more complicated when including bulk operators. Indeed, whenever a bulk operator is included, one has to include its distance r from the defect – essentially its image on the other side of the defect. For example, the one-point function of an operator in the bulk no longer vanishes (it can always be made to vanish in a regular QFT using $\mathcal{O} \to \mathcal{O} + b$, which amounts to adding a counter-term). In the linear defect case, it can be viewed as the two-point function of the bulk operator with the line. In the $p > 1$ case, it can be viewed as the two-point function of the operator with its image by the defect. The following results can be obtained with a defect (see classic reference [18], or lectures [15]):

$$\langle \mathcal{O}(x)\rangle = \frac{a_{\mathcal{O}}}{|x|_{d-p}^{\Delta}}, \quad \langle \mathcal{O}_1(x)\hat{\mathcal{O}}_2(0)\rangle = \frac{C_{12}}{|x|_{d-p}^{\Delta_1-\Delta_2}|x|^{2\Delta_2}} \tag{1.19}$$

where $|x|_{d-p}$ is the norm of the projection of the distance vector along the subspace orthogonal to the defect. Note that as mentioned before, the above one-point correlator behaves as a two-point function, while the two-point behaves as a three-point (because of the image of the bulk operator on the other side of the defect). Thus, two-point functions of bulk operators behave like four-point functions, and as such don't have simple ansatze.

In the following work, we will make sure that we're working with unitary theories, in keeping with conventional QFT. It turns out that unitarity heavily constrains values of scaling dimensions. For a scalar field ϕ, unitarity requires (see [15]):

$$\Delta_\phi \geq \max\left(0, \frac{d-2}{2}\right) \tag{1.20}$$

The last ingredient from defect CFT (DCFT) worth mentioning for the purpose of this work is the defect free energy $\mathcal{F}$, or the g-function in some cases. The free energy is defined using the quotient of the partition function with a defect by the partition function without one:

$$\mathcal{F} := -\log \frac{Z_{\text{defect}}}{Z_0} \tag{1.21}$$

This expression is incidentally reminiscent of the definitions of free energy in statistical mechanics. One often also looks at the g-function defined by $g = Z_{\text{defect}}/Z_0$. These functions are used in an equivalent of the second law of thermodynamics for renormalization group flows. It has been shown [19, 20] that for line defects and two-dimensional spherical defects, g always decreases along an RG flow until reaching a fixed point. In higher dimensional cases, one looks at coefficients appearing in $\mathcal{F}$ rather than g, and this is referred to as an F-theorem [21, 22].

1.3 The long-range $O(N)$ model

This work focuses to a large extent on the long-range Ising model and its vector extension in dimension $2 < d < 4$. The long-range Ising model can be defined on a spin lattice through the following Hamiltonian:

$$H = -J \sum_{i \neq j} \frac{s_i s_j}{|i-j|^{d+\sigma}} \tag{1.22}$$

where we take $J > 0$ to stick to the ferromagnetic case. This is slightly closer to reality than the Ising model in classical electrodynamics for example, since magnetic moments interact via a power-law with all of the other moments in space. This model undergoes a second order phase transition at a certain temperature $T = T_c$ (proved in $d = 1$ in for some values of σ in [23, 24], for $d \geq 2$ in [25] using a generalization of Peierl's argument). In accordance with the Landau-Ginzburg approach, at $T = T_c$, one can replace it with the following continuum model of a real scalar field:

$$S = \frac{\mathcal{N}_\sigma}{2} \int d^d x d^d y \frac{\phi(x)\phi(y)}{|x-y|^{d+\sigma}} + \frac{\lambda_0}{4!} \int d^d x \phi(x)^4 \tag{1.23}$$

The validity of this approach can be motivated by the fact that the microscopic details of the lattice at $T = T_c$ can be ignored, the characteristics of critical theories being universal. Henceforth, we define the above action as being that of the long-range Ising model, and forget about its discrete statistical mechanical counterpart entirely.

There is another way of getting to the action in equation (1.23), consistent with the AdS/CFT picture. Indeed, if one considers a massive field $\Phi(x, y)$ whose action is

$$S = \frac{1}{2} \int_\Omega d^d x dy \sqrt{g} \left[\partial_M \Phi \partial^M \Phi + m^2 \Phi^2\right] + \frac{\lambda}{4!} \int_{\partial\Omega} d^d x \Phi^4 \tag{1.24}$$

where g_{MN} is the metric of the Poincaré folliation of AdS_{d+1}, then integrating out the bulk field $\Phi(x, y > 0)$ leads to a theory for $\phi(x) = \Phi(x, 0)$ that is identical to that of the LRI. In other words, the LRI, defined in flat space, is dual to a theory of a massive field in a bulk AdS space (of constant negative curvature) [13, 26, 27]. It turns out that considering higher dimensional spaces on which the LRI is a defect theory has proven fruitful for proving that the LRI fixed point is a CFT [13].

Traditional theories have Lagrangians which are local i.e. they only depend on fields $\phi(x)$ at a given position x. Formally, one defines a local theory as a theory whose action is writen as a single integral over space of a Lagrangian depending on derivatives of fields up to a finite order. Otherwise, a theory is said to be nonlocal (the LRI is in fact strongly nonlocal following the criteria in section 2 of [28]). One way to define a nonlocal Lagrangian for a free theory is to start with a Gaussian theory. Keeping notation from section 1.1, local theories have $D(x, y) \propto \delta(x-y)$. Removing this δ-function leads to a non-local kinetic term. A natural way to extend usual Klein-Gordon type terms with $D(x, y) \propto -\delta(x-y)\partial^2$ is to replace $-\partial^2$ with $\mathcal{L}_\sigma = (-\partial^2)^{\sigma/2}$ for $\sigma \in \mathbb{R}$. The operator $\mathcal{L}_\sigma$ is called the fractional Laplacian. We define it here in such a way that $\mathcal{L}_\sigma e^{ipx} = |p|^\sigma e^{ipx}$. In position space, it can be defined by:

$$\mathcal{L}_\sigma \phi(x) = \frac{2^\sigma \Gamma\left(\frac{d+\sigma}{2}\right)}{\pi^{\frac{d}{2}} \Gamma\left(-\frac{\sigma}{2}\right)} \int d^d y \frac{\phi(y)}{|x-y|^{d+\sigma}} \tag{1.25}$$

Denoting the normalization $\mathcal{N}_\sigma$, we now define the interacting long-range $O(N)$ model, which is a natural extension of the LRI.

$$S = \frac{\mathcal{N}_\sigma}{2} \int d^d x d^d y \frac{\phi_a(x)\phi_a(y)}{|x-y|^{d+\sigma}} + \frac{\lambda_0}{4!} \int d^d x \left(\phi_a(x)^2\right)^2 \tag{1.26}$$

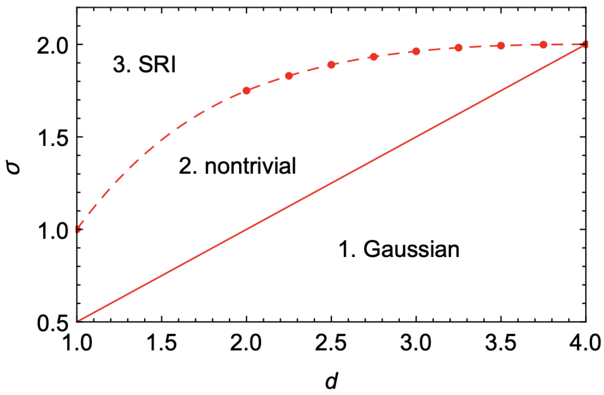


Figure 1: Phase diagram of the long-range Ising model critical theory, taken from [13].

where $a \in [\![1, N]\!]$ is a dummy variable implicitly summed over.[4] The case $N = 1$ reduces to the long-range Ising model (1.23) extensively studied in [13, 29, 30]. This theory exhibits different behaviors depending on the value of σ. For $\sigma < d/2$, the quartic interaction is irrelevant, hence the critical theory is free. For $\sigma > \sigma_* = 2 - \eta_{\mathrm{SRI}}$, the critical theory is the short-range Ising model [5]. For $d/2 < \sigma < \sigma_*$, the critical theory is a non-trivial non-Gaussian one, the quartic interaction being relevant and making the theory flow to an IR fixed point, the so-called LRI fixed point (see figure 1). In this work, we often study the long-range $O(N)$ model close to this crossover and we generalize to $N > 1$ since this is quite straightforward.

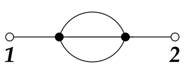
The propagator of the long-range $O(N)$ model can easily be derived using Schwinger parametrization (see appendix B for proof). It is given by:

$$G_{ab}(x) = \frac{2^{d-\sigma}\Gamma\left(\frac{d-\sigma}{2}\right)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{\sigma}{2}\right)}\frac{\delta_{ab}}{|x|^{d-\sigma}} = \frac{N_\phi \delta_{ab}}{|x|^{2\Delta_\phi}} \tag{1.27}$$

where it's worth keeping in mind that we've conventionally chosen propagators normalized in momentum space [6]. In order to study the LRI fixed point, one conducts an ε-expansion, taking $\sigma = (d+\varepsilon)/2$ in order to make the ϕ^4 interaction marginally relevant. Since we're looking at massless (scaleless) theories, one can throw away tadpole diagrams, and one can cancel all divergences by shifting the value of the λ-coupling or renormalizing the field ϕ_a. We thus introduce $\phi_a = Z_\phi[\phi_a]$, $\lambda_0 = Z_\lambda \lambda \mu^\varepsilon$ and a mass scale μ. Thus the full action we are looking at is:

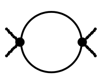
$$S = \frac{\mathcal{N}_\sigma}{2}\int d^dx d^dy Z_\phi^2 \frac{[\phi_a](x)[\phi_a](y)}{|x-y|^{d+\sigma}} + \frac{Z_\lambda \lambda \mu^\varepsilon}{4!}\int d^dx \left(Z_\phi^2[\phi_a](x)^2\right)^2 \tag{1.28}$$

and we tune the coefficients in Z_ϕ and Z_λ so as to cancel the poles in ε in correlators of the renormalized fields. One can derive Z_ϕ by making the two-point function $\langle[\phi](x)[\phi](0)\rangle$ finite. The first non-trivial diagram contributing to this correlator is at second order in λ. It is given by the following integral, easily computable using the master integral (C.6) in the appendix and the symmetry factors found in chapter 6 of [9]:

$$\frac{N+2}{3}\frac{1}{6}$$


$$= \frac{N+2}{18}\frac{N_\sigma^5 \lambda^2 \mu^{2\varepsilon} w^{(d)}_{6\Delta_\phi}\left[w^{(d)}_{2\Delta_\phi}\right]^2}{w^{(d)}_{10\Delta_\phi - 2d}}\frac{1}{|x|^{10\Delta_\phi - 2d}} = \lambda^2 \frac{(N+2)\Gamma\left(-\frac{d}{4}\right)}{18\, 2^d (2\pi)^{\frac{3d}{2}}\Gamma\left(\frac{3d}{4}\right)}\frac{1}{|x|^{\frac{d}{2}}} + \mathcal{O}(\varepsilon) \tag{1.29}$$

This quantity doesn't diverge for $2 < d < 4$, hence there is no wavefunction renormalization and ϕ doesn't gain an anomalous dimension in the IR[7]. Next, we turn to the renormalization of the coupling. This involves the following diagrams, which we compute in momentum space for "incoming" total momentum k:

$$\frac{N+8}{9}\frac{3}{2}$$


$$= \frac{(N+8)\lambda^2\mu^{2\varepsilon}}{6\,(2\pi)^d}\frac{\left[w^{(d)}_\sigma\right]^2}{w^{(d)}_{2\sigma-d}}\frac{1}{k^{2\sigma-d}} = \frac{(N+8)\lambda^2}{3(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)\varepsilon} + \mathcal{O}(1) \tag{1.30}$$

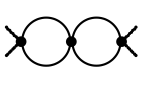
[4] We won't always specify the index, especially when the context is clear.

[5] This statement is less trivial, and is motivated in [13]. We also provide some justification of the short-range nature of this phase in section 2.3.

[6] Sometimes authors choose to normalize in position space, as is the case in [13].

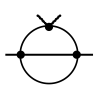
[7] In fact, nonlocal theories should *not* exhibit wavefunction renormalization. We discuss this further in section 2.3.

$$\frac{N^2+6N+30}{27}\,\frac{3}{4}$$



$$\begin{aligned}
&= -\frac{(N^2+6N+30)\lambda^3\mu^{3\varepsilon}}{36\,(2\pi)^{2d}}\,\frac{\left[w_\sigma^{(d)}\right]^4}{\left[w_{2\sigma-d}^{(d)}\right]^2}\,\frac{1}{k^{4\sigma-2d}} \\
&= -\frac{(N^2+6N+30)\lambda^3}{9(4\pi)^d\Gamma\left(\frac{d}{2}\right)^2}\left(\frac{1}{\varepsilon^2}-\frac{\gamma+2\ln k-3\ln\mu+2\psi\left(\frac{d}{4}\right)-\psi\left(\frac{d}{2}\right)}{\varepsilon}\right)+\mathcal{O}(1)
\end{aligned} \tag{1.31}$$

$$\frac{5N+22}{27}\cdot 3$$



$$\begin{aligned}
&\supset -\frac{(5N+22)\lambda^3\mu^{3\varepsilon}}{9\,(2\pi)^d\,(4\pi)^{\frac{d}{2}}}\,\frac{\left[w_\sigma^{(d)}\right]^2}{w_{2\sigma-d}^{(d)}}\,\frac{\Gamma\left(2\sigma-d\right)}{\Gamma\left(\frac{\sigma}{2}\right)^2}\,\frac{\Gamma\left(d-\frac{3\sigma}{2}\right)^2}{\Gamma\left(2d-3\sigma\right)}\,\frac{\Gamma\left(\sigma\right)}{\Gamma\left(2\sigma-\frac{d}{2}\right)}\,\frac{1}{k^{4\sigma-2d}} \\
&= -\frac{(5N+22)\lambda^3}{9(4\pi)^d\Gamma\left(\frac{d}{2}\right)^2}\left(\frac{2}{\varepsilon^2}-\frac{3\gamma+4\ln k-6\ln\mu+6\psi\left(\frac{d}{4}\right)-3\psi\left(\frac{d}{2}\right)}{\varepsilon}\right)+\mathcal{O}(1)
\end{aligned} \tag{1.32}$$

The first two diagrams are easy to evaluate using the master integral (C.6). The last one is much more involved, and rather than calculating it exactly, we extract its poles in ε using a method outlined in [9] and reviewed in appendix E. To two-loop order, we introduce the coupling renormalization factor Z_λ to cancel out the poles in ε above:

$$Z_\lambda = 1+\frac{N+8}{3\Gamma\left(\frac{d}{2}\right)\varepsilon}\frac{\lambda}{(4\pi)^{\frac{d}{2}}}+\frac{1}{9\Gamma\left(\frac{d}{2}\right)^2}\frac{\lambda^2}{(4\pi)^d}\left(\frac{N+8}{\varepsilon^2}-\frac{(5N+22)\left(\psi\left(\frac{d}{2}\right)-2\psi\left(\frac{d}{4}\right)-\gamma\right)}{\varepsilon}\right) \tag{1.33}$$

where ψ is the logarithmic derivative of the Γ-function, known as the *digamma* function. We finally arrive at the following β-function[8], which is to be compared with expressions found in [31, 32]:

$$\beta_\lambda = -\varepsilon\lambda+\frac{N+8}{3\Gamma\left(\frac{d}{2}\right)}\frac{\lambda^2}{(4\pi)^{\frac{d}{2}}}-\frac{2(5N+22)(\psi\left(\frac{d}{2}\right)-2\psi\left(\frac{d}{4}\right)-\gamma)}{9\Gamma\left(\frac{d}{2}\right)^2}\frac{\lambda^3}{(4\pi)^d}+\mathcal{O}\left(\frac{\lambda^4}{(4\pi)^6}\right) \tag{1.34}$$

This β-function has a non-trivial zero, or fixed point for this RG flow:

$$\frac{\lambda_*}{\Gamma\left(\frac{d}{2}\right)(4\pi)^{\frac{d}{2}}} = \frac{3}{N+8}\varepsilon+\frac{6\,(5N+22)\left(\psi\left(\frac{d}{2}\right)-2\psi\left(\frac{d}{4}\right)-\gamma\right)}{(N+8)^3}\varepsilon^2+\mathcal{O}\left(\varepsilon^2\right) \tag{1.35}$$

It is at this point that the LRI is extended to the long-range $O(N)$ model, and it can be shown to be conformal by extending arguments first formulated in [13] and reviewed in [29, 30].

1.4 Existence and classification of some non-trivial defects

Now that we have reviewed the relevant elements of QFT and CFT, and introduced some of the models of interest in the bulk, it is time to turn to the main subject of this work: constructing non-trivial defects in nonlocal $O(N)$ invariant theories, mainly the long-range $O(N)$ model. However, it is instructive to first examine whether there can be non-trivial defects in the first place. Indeed, it has been shown that in integer dimensions less than four, local theories which are free in the bulk cannot have non-trivial defects [6] – trivial meaning Gaussian. There are two loopholes to this statement which are exemplified in this work.

- **Bulk interactions**: The long-range $O(N)$ model as we've presented it includes bulk ϕ^4 interactions. Hence, the bulk theory isn't free and the above statement no longer applies. This will be the case for example in the localized magnetic field examined in section 2.
- **Non-locality**: Let's forget bulk interactions for the moment and merely look at the kinetic part of the long-range $O(N)$ model. It turns out that it can be rendered local in a higher dimensional bulk of non-integer dimension $d+2-\sigma$ [13, 33]. This dimension being non-integer, there *can* be non-trivial defects in the free theory a priori (see section 4).

Let us briefly classify some of the defects which might yield interesting results. The first case we look at is a defect interaction going like a power of the LRI field on a p-dimensional submanifold:

$$S = S_0+h_0\int_p d^px\,\phi_1(x)^n \tag{1.36}$$

Making this interaction classically marginal while keeping bulk interactions marginal requires $nd=4p$. Let's look at different cases depending on the bulk dimension $d\in[3,4]$.

[8] It's worth noting that this β-function yields a CFT datum for free (see appendix F for proof of the general statement): $\Delta_{\phi^4} = d+\left.\frac{\partial\beta_\lambda}{\partial\lambda}\right|_{\lambda=\lambda_*} = d+\varepsilon-\frac{2(5N+22)(\psi(\frac{d}{2})-2\psi(\frac{d}{4})-\gamma)}{(N+8)^2}\varepsilon^2+\mathcal{O}\left(\varepsilon^3\right)$

- $\boldsymbol{d=3}$: 4 and 3 being coprime, the requirement $3n = 4p$ leads to $3|p$ i.e. $p = 3$. This is a trivial defect and therefore not worth studying.
- $\boldsymbol{d=4}$: $(p,n) = (1,1),(2,2),(3,3)$ are all viable options a priori.

Therefore, contrary to the case of the defectless long-range $O(N)$ model, we are here forced to look at an ε-expansion near $d = 4$. In this work, we'll look at the ε-expansion for $(p,n) = (1,1)$ and $(2,2)$ i.e. the localized magnetic field and the surface defect.

2 Localized magnetic field in the long-range $O(N)$ model

The localized magnetic field was extensively studied in [1, 2] for a local $O(N)$-symmetric free field. In this section, we generalize those results to the nonlocal, long-range cousin of the $O(N)$ model. We therefore study the following action:

$$S = \frac{\mathcal{N}_\sigma}{2}\int d^dx d^dy \frac{\phi_a(x)\phi_a(y)}{|x-y|^{d+\sigma}} + \frac{\lambda_0}{4!}\int d^dx\left(\phi_a(x)^2\right)^2 + h_0\int dx\phi_1(x) \tag{2.1}$$

2.1 ε-expansion

For h to renormalize, just like for the local $O(N)$ model one still needs to approach $d = 4$. For $\varepsilon > 0$, let's set $d = 4-\varepsilon$ and $\sigma = (d+\kappa\varepsilon)/2 = 2-(1-\kappa)\varepsilon/2$. This leads to the the scaling dimension $\Delta_\phi = 1-\varepsilon(1+\kappa)/4$ for ϕ_a, while the mass dimension of the defect coupling becomes $[h] = \varepsilon(1+\kappa)/4$. The bulk coupling λ still undergoes renormalization via the same diagrams (four-point function). However, one needs to recalculate the various renormalization factors using the new scheme defined above. It turns out that ϕ_a now renormalizes at two-loop order:

$$Z_\phi = 1 - \frac{(N+2)\lambda^2}{18(4\pi)^4(1+3\kappa)\varepsilon} + \mathcal{O}\left(\frac{\lambda^3}{(4\pi)^6}\right) \tag{2.2}$$

The four-point function has similar poles to the defectless case in dimension $d < 4$, however Z_λ has a distinct expression, in part because we no longer have $Z_\phi \neq 1$ at this order:

$$Z_\lambda = 1 + \frac{N+8}{3\kappa\varepsilon}\frac{\lambda}{(4\pi)^2} + \left[\frac{(N+8)^2}{9\kappa^2\varepsilon^2} - \frac{5N+22+\kappa(13N+62)}{9(1+3\kappa)\kappa\varepsilon}\right]\frac{\lambda^2}{(4\pi)^4} + \mathcal{O}\left(\frac{\lambda^3}{(4\pi)^6}\right) \tag{2.3}$$

This leads to the following β-function:

$$\beta_\lambda = -\kappa\varepsilon\lambda + \frac{N+8}{3}\frac{\lambda^2}{(4\pi)^2} - \frac{10N+44+\kappa(26N+124)}{9(1+3\kappa)}\frac{\lambda^3}{(4\pi)^4} + \mathcal{O}\left(\frac{\lambda^4}{(4\pi)^6}\right) \tag{2.4}$$

which has a non-trivial (Wilson-Fisher) fixed point [9] at:

$$\frac{\lambda_*}{(4\pi)^2} = \frac{3}{N+8}\kappa\varepsilon + \frac{6}{(N+8)^3}\frac{5N+22+\kappa(13N+62)}{1+3\kappa}\kappa^2\varepsilon^2 + \mathcal{O}\left(\varepsilon^2\right) \tag{2.5}$$

These computations allow one to derive the anomalous dimension γ_ϕ of ϕ_a at the IR fixed point. Indeed:

$$\gamma_\phi = \beta_\lambda\frac{\partial\ln Z_\phi}{\partial\lambda}\bigg|_{\lambda=\lambda_*} = \frac{\kappa^3(N+2)}{(1+3\kappa)(N+8)^2}\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right) \tag{2.6}$$

and at the fixed point, ϕ will have a scaling dimension $\Delta_\phi + \gamma_\phi$.[10]

As a first consistency check, one can verify that at $\kappa = 1$, one retrieves the expressions in [2]. Another thing worth checking, which might be more obvious in flows which don't depend on an extra parameter like κ, is the (numerical) stability of the bulk IR fixed point we found above. This amounts to studying the sign of the derivative of β_λ at λ_*:

$$\beta'_\lambda(\lambda_*) = \kappa\varepsilon - \frac{2\left(5N+22+\kappa(13N+62)\right)}{(1+3\kappa)(N+8)^2}\kappa^2\varepsilon^2 + \mathcal{O}(\varepsilon^3) \tag{2.7}$$

This leads to the following "numerical" consistency condition on how small ε must be:

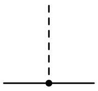
$$\varepsilon < \varepsilon_{\text{thresh}} = \frac{(1+3\kappa)(N+8)^2}{2\kappa\left(5N+22+\kappa(13N+62)\right)} \tag{2.8}$$

[9] Moreover, one can easily derive another fixed point observable using a derivative of the β-function computed above: $\Delta_{\phi^4} = 4-(1-\kappa)\varepsilon - \frac{2(5N+22+\kappa(13N+62))}{(1+3\kappa)(N+8)^2}\kappa^2\varepsilon^2 + \mathcal{O}(\varepsilon^3)$

[10] This statement is actually subtle, since nonlocal theories shouldn't exhibit wavefunction renormalization for ϕ. We further discuss this in section 2.3.

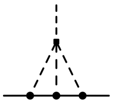
From the above inequality, one can then numerically study for $\kappa \in [0,1]$ the minimum value of the dimension $d(\kappa)$ such that the fixed point is stable. As $\kappa \to 0$ from $\kappa = 1$, one can check numerically that d_{thresh} goes from 2.4 to $-\infty$. In other words, stability is never an issue for reasonable values of (κ, ε) (for example the fixed point $d = 3$ is always stable to two-loop order). Note that this result is perhaps less trustworthy for larger values of ε, which might require perturbation theory to higher order.

The magnetic coupling h also renormalizes as implied by the requirement of a non-trivial behavior. The renormalization of h is due to the following diagrams, which can be systematically evaluated using master integrals in the appendix.

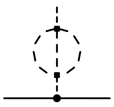


$$= -N_\sigma h_0 \frac{\sqrt{\pi}\Gamma\left(\Delta_\phi - \frac{1}{2}\right)}{\Gamma(\Delta_\phi)} \frac{1}{|x|_{d-1}^{2\Delta_\phi - 1}} \tag{2.9}$$

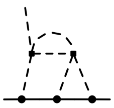
[11]



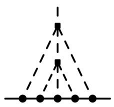
$$= \frac{N_\sigma^4 \lambda_0 h_0^3}{6} \left[\frac{\sqrt{\pi}\Gamma\left(\Delta_\phi - \frac{1}{2}\right)}{\Gamma(\Delta_\phi)}\right]^4 \frac{w_{2\Delta_\phi-1}^{(d-1)} w_{6\Delta_\phi-3}^{(d-1)}}{w_{8\Delta_\phi-3-d}^{(d-1)}} \frac{1}{|x|_{d-1}^{8\Delta_\phi-d-3}} \tag{2.10}$$



$$= -\frac{(N+2)N_\sigma^5 \lambda_0^2 h_0}{18} \frac{\pi^{\frac{3}{2}}\Gamma\left(\Delta_\phi - \frac{1}{2}\right)^2 \Gamma\left(3\Delta_\phi - \frac{1}{2}\right)}{\Gamma(\Delta_\phi)^2\Gamma(3\Delta_\phi)} \frac{w_{6\Delta_\phi-1}^{(d-1)} \left[w_{2\Delta_\phi-1}^{(d-1)}\right]^2}{w_{10\Delta_\phi-2d-1}^{(d-1)}} \frac{1}{|x|_{d-1}^{10\Delta_\phi-2d-1}} \tag{2.11}$$



$$= -\frac{(N+8)N_\sigma^6 \lambda_0^2 h_0^3}{36} \frac{\Gamma\left(\Delta_\phi - \frac{1}{2}\right)^4 \pi^{\frac{5}{2}}\Gamma\left(2\Delta_\phi - \frac{1}{2}\right)}{\Gamma(\Delta_\phi)^4\Gamma(2\Delta_\phi)} \frac{w_{4\Delta_\phi-1}^{(d-1)} w_{4\Delta_\phi-2}^{(d-1)} w_{2\Delta_\phi-1}^{(d-1)} w_{10\Delta_\phi-d-3}^{(d-1)}}{w_{8\Delta_\phi-d-2}^{(d-1)} w_{12\Delta_\phi-2d-3}^{(d-1)}} \frac{1}{|x|_{d-1}^{12\Delta_\phi-2d-3}} \tag{2.12}$$



$$= -\frac{N_\sigma^7 \lambda_0^2 h_0^5}{12} \left[\frac{\sqrt{\pi}\Gamma\left(\Delta_\phi - \frac{1}{2}\right)}{\Gamma(\Delta_\phi)}\right]^7 \frac{\left[w_{2\Delta_\phi-1}^{(d-1)}\right]^2 w_{6\Delta_\phi-3}^{(d-1)} w_{12\Delta_\phi-d-5}^{(d-1)}}{w_{8\Delta_\phi-d-3}^{(d-1)} w_{14\Delta_\phi-2d-5}^{(d-1)}} \frac{1}{|x|_{d-1}^{14\Delta_\phi-2d-5}} \tag{2.13}$$

Renormalizing these diagrams leads to the following renormalization factor Z_h for the h coupling:

$$\begin{aligned} Z_h = 1 + \frac{h^2}{3(1+3\kappa)\varepsilon} \frac{\lambda}{(4\pi)^2} + &\left[\frac{(N+2)h}{18(1+3\kappa)\varepsilon} + \left(\frac{1}{\varepsilon^2} - \frac{1+3\kappa^2 - (1-\kappa)(\gamma - \ln 4\pi)}{4\kappa\varepsilon}\right) \frac{2(N+8)h^2}{9(1+3\kappa)(1+5\kappa)}\right. \\ &\left. + \left(\frac{2}{(1+3\kappa)\varepsilon^2} - \frac{1}{\varepsilon}\right) \frac{h^4}{12(1+3\kappa)}\right] \frac{\lambda^2}{(4\pi)^4} + \mathcal{O}\left(\frac{\lambda^3}{(4\pi)^6}\right) \end{aligned} \tag{2.14}$$

This subsequently yields the following β-function:

$$\begin{aligned} \beta_h = -\frac{\varepsilon(1+\kappa)}{4} h + \frac{\lambda}{(4\pi)^2} \frac{h^3}{6} + \frac{\lambda^2}{(4\pi)^4} \left(\frac{(N+2)\kappa h}{9(1+3\kappa)} - \frac{(N+8)\left(1+3\kappa^2 - (1-\kappa)(\gamma - \ln(4\pi))\right)}{36\kappa(1+3\kappa)} h^3 - \frac{h^5}{12}\right) \\ + \mathcal{O}\left(\frac{\lambda^3}{(4\pi)^6}\right) \end{aligned} \tag{2.15}$$

which reduces to that derived in [1, 2] in the limit $\kappa \to 1$. Next, let's derive this β-function at the bulk fixed point. It is given by:

$$\begin{aligned} &\beta_h(\lambda_*) = -\left(\frac{(1+\kappa)h}{4} - \frac{\kappa h^3}{2(N+8)}\right)\varepsilon \\ &+ \left(\frac{(N+2)\kappa^3 h}{(1+3\kappa)(N+8)^2} + \frac{\left[4\kappa(5N+22+\kappa(13N+62)) - (N+8)^2\left(1+3\kappa^2 + (1-\kappa)(\ln(4\pi) - \gamma)\right)\right]\kappa h^3}{4(1+3\kappa)(N+8)^3}\right. \\ &\left. - \frac{3\kappa^2 h^5}{4(N+8)^2}\right)\varepsilon^2 + \mathcal{O}(\varepsilon^3) \end{aligned} \tag{2.16}$$

A non trivial defect fixed point can then be derived perturbatively in ε:

[11] The diagrams in this section are taken from [2].

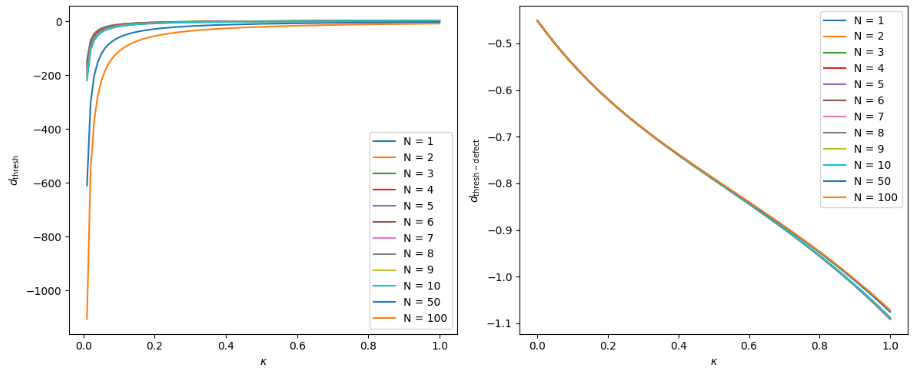


Figure 2: Threshold dimension under which the IR fixed point is no longer stable as a function of the parameter κ. On the left, the threshold criterion derived from the bulk β-function. On the right, the criterion for the defect β-function.

$$
\begin{aligned}
h_*^2 &= \frac{(N+8)(1+\kappa)}{2\kappa} + \frac{\left(5+2(1-\kappa^2)(\ln(4\pi)-\gamma)\right)(N+8)}{8\kappa(1+3\kappa)}\varepsilon \\
&+ \frac{(15\kappa^2+27\kappa+17)N^2+8(15\kappa^2+36\kappa+29)N+48(9\kappa^2+22\kappa+19)}{8(1+3\kappa)(N+8)}\varepsilon + \mathcal{O}(\varepsilon^2)
\end{aligned}
\tag{2.17}
$$

As before, one can also study the stability of the fixed point. Requiring a positive derivative of the β-function at the fixed point yields the following numerical requirement:

$$
\frac{1+\kappa}{2} \geq \left(\frac{(9N^2+160N+608)\kappa^3}{8(1+3\kappa)(N+8)^2} - \frac{3}{8}\frac{7\kappa^2+5\kappa+1+\left(1-\kappa^2\right)(\ln 4\pi-\gamma)}{1+3\kappa} \right)\varepsilon \tag{2.18}
$$

Incidentally, this implies that the scaling dimension of ϕ on the defect is positive (see next section), hence satisfying a unitarity bound.

2.2 Some defect observables

Scaling dimension. We can now derive the defect scaling dimension of ϕ by taking a derivative of the β-function for h:

$$
\Delta_{\hat{\phi}} = 1 + \frac{1+\kappa}{2}\varepsilon - \frac{(9N^2+160N+608)\kappa^3}{8(1+3\kappa)(N+8)^2}\varepsilon^2 - \frac{3}{8}\frac{7\kappa^2+5\kappa+1+\left(1-\kappa^2\right)(\ln 4\pi-\gamma)}{1+3\kappa}\varepsilon^2 + \mathcal{O}(\varepsilon^3) \tag{2.19}
$$

One-point data. Now that we've renormalized the localized magnetic field, let's plug the renormalization factors back into the relevant Feynman diagrams. The convention is to define the defect CFT datum a_ϕ such that:

$$
\langle [\phi_a(x)] \rangle = \delta_{a,1} \frac{\sqrt{N_\phi}a_\phi}{|x|_{d-p}^{\Delta_\phi+\gamma_\phi}} \tag{2.20}
$$

Hence:

$$
\begin{aligned}
a_\phi^2 = \frac{(\kappa+1)(N+8)}{8\kappa} &+ \Bigg[\frac{3\kappa^2\left(N^2(3+\log(4))+8N(1+\log(16))+16(1+\log(256))\right)}{32(3\kappa+1)(N+8)} \\
&+ \frac{\kappa\left(9N^2+14(N+8)^2\log(2)-96\right)}{32(3\kappa+1)(N+8)} + \frac{10(N+8)^2\log(2)-N(N+56)-240}{32(3\kappa+1)(N+8)} \\
&+ \frac{(N+8)(\log(4)-1)}{32\kappa(3\kappa+1)} \Bigg] \varepsilon + \mathcal{O}\left(\varepsilon^2\right)
\end{aligned}
\tag{2.21}
$$

And when $\kappa \to 1$ we recover the expansion cited in [2]. Hence in the IR the $O(N)$ internal symmetry is broken and the DCFT is instead $O(N-1)$-invariant.

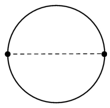
g-function. In order to study the RG flow from the UV to the IR, it is useful to calculate the g-function of this DCFT, and verify that the g-theorem holds [19]. To begin with, let's look at the theory without a ϕ^4 coupling.

$$\begin{aligned} Z_{\text{defect}} &= \int \mathcal{D}\phi e^{-S_0} e^{-h_0 \int \phi} = Z_0 \sum_{k=0}^{+\infty} \frac{(-h_0)^{2k}}{(2k)!} \int dx_1 \dots dx_{2k} \langle \phi_1(x_1) \dots \phi_1(x_{2k}) \rangle_0 \\ &= Z_0 \sum_{k=0}^{+\infty} \frac{(-h_0)^{2k}}{(2k)!} N_\sigma^k \frac{(2k)!}{2^k k!} \left(4^{1-\Delta_\phi} R^{2-2\Delta_\phi} \pi \frac{\Gamma\left(\frac{1}{2} - \Delta_\phi\right)\sqrt{\pi}}{\Gamma(1-\Delta_\phi)} \right)^k \end{aligned} \tag{2.22}$$

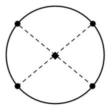
where to get from the second to the third line, we used Wick's theorem and the fact that for a string of $2k$ fields, there are $(2k)!/(2^k k!)$ Wick contractions. Recognizing the series representation of the exponential, we arrive at:

$$\log g = \log \frac{Z_{\text{defect}}}{Z_0} = \pi h^2 2^{1-2\Delta_\phi} R^{2-2\Delta_\phi} N_\phi \frac{\Gamma\left(\frac{1}{2} - \Delta_\phi\right)\sqrt{\pi}}{\Gamma(1-\Delta_\phi)} \tag{2.23}$$

Next, let's look at the case with ϕ^4 interactions in the bulk. This is no longer exactly solvable, and we have to resort to Feynman diagrams.



$$= N_\sigma 2^{1-2\Delta_\phi} h_0^2 R^{2-2\Delta_\phi} \pi \frac{\Gamma\left(\frac{1}{2} - \Delta_\phi\right)\sqrt{\pi}}{\Gamma(1-\Delta_\phi)} = -\frac{1+\kappa}{16} h_0^2 \varepsilon + \mathcal{O}(\varepsilon^2) \tag{2.24}$$



$$= -\frac{\lambda_0 h_0^4}{384\pi^2} + \mathcal{O}(\varepsilon) \tag{2.25}$$

where the second diagram is evaluated using appendix D. Using the relations between bare and renormalized couplings, we arrive at

$$\log g = -\frac{(1+\kappa)\varepsilon}{16} h^2 + \frac{\kappa h^4 \lambda_*}{192\pi^2(3\kappa+1)} + \mathcal{O}\left(\lambda_*^2\right) \tag{2.26}$$

and using the value of the defect fixed point coupling h_* previously derived:

$$\log g_{\text{IR}} = -\frac{(\kappa+1)^3(N+8)}{32\kappa(3\kappa+1)} \varepsilon + \mathcal{O}\left(\varepsilon^2\right) < 0 = \log g_{\text{UV}} \tag{2.27}$$

Hence the g-theorem holds true perturbatively.

2.3 Nonlocality and wavefunction renormalization

Let's now turn to a curiosity which arose in the renormalization of the bulk in $d = 4-\varepsilon$. First of all, nonlocal theories do not exhibit wavefunction renormalization. This is essentially because the divergences are local, while the kinetic part of the Lagrangian is nonlocal [13]. However, this seems to contradict results above, where we saw that the field does indeed renormalize close to $d = 4$ and $\sigma = 2$. This can be explained by the fact that at $d = 4$, the LRI fixed point described the theory of a local field (one can look at the scaling dimensions for example). Hence, expanding close to $d = 4$ and $\sigma = 2$ amounts to perturbing a local field with non local interactions. Indeed, let's look at the kinetic part of the Lagrangian of the LRI in momentum space:

$$\mathcal{L}_{\text{GFF}} \propto \int d^d p \hat{\phi}(p) |p|^{2-\frac{1-\kappa}{2}\varepsilon} \hat{\phi}(-p) = \int d^d p \hat{\phi}(p) |p|^2 \hat{\phi}(-p) - \frac{1-\kappa}{2} \varepsilon \int d^d p \hat{\phi}(p) |p|^2 \ln|p| \hat{\phi}(-p) + \mathcal{O}\left(\varepsilon^2\right) \tag{2.28}$$

The first term yields the kinetic part of a local field theory, while the second yields an unusual interaction term, which can be conjectured to be nonlocal. Incidentally, if we move in the $\kappa = 1$ direction, the theory is local up to $\mathcal{O}(\varepsilon^2)$, which is in line with the picture in figure 1. However, we know that in the intermediate zone in the phase diagram of the LRI (the σ-d graph) the theory is nonlocal. Hence, we can impose that our trajectories do not contribute an anomalous dimension to ϕ. To do so, we can impose

$$\Delta^{(0)}(\kappa) + \gamma(\lambda_*) = \Delta^{(0)}(\kappa_1) \tag{2.29}$$

This entails

$$\kappa_1 = \kappa - \frac{4\kappa^3(N+2)}{(1+3\kappa)(N+8)^2} \varepsilon \tag{2.30}$$

and the trajectories described by (κ_1, ε) are then given by

$$\sigma(d) = 2 + \frac{(1-\kappa)^2 (1+3\kappa)(N+8)^2}{32\kappa^3 (N+2)} - \frac{2\kappa^3 (N+2)}{(1+3\kappa)(N+8)^2} \left(4 - d + \frac{(1-\kappa)(1+3\kappa)(N+8)^2}{8\kappa^3 (N+2)} \right)^2 \tag{2.31}$$

It's worth noting that for $\kappa = 1$, i.e. if we start heading to the zone where the theory is meant to be the short-range theory, then

$$\sigma(d) = 2 - \frac{(N+2)}{2(N+8)^2}(4-d)^2 = 2 - \eta_{\mathrm{SRI}} \tag{2.32}$$

In other words, the "correct" trajectory describing a long-range theory (without an anomalous dimension) espouses the boundary curve between the upper short-range region and the lower long-range region. Clearly, the above formulae are valid for d close enough to 4. This is exemplified by the region close to $d = 4$ swept by $\kappa \in [0,1]$ in figure 3, which looks identical in both the graphs on the left and on the right. One might expect that the formula for $\kappa(\varepsilon)$ at arbitrary order in ε might reproduce the entire 'nontrivial' phase between the LRI line and the short-range crossover curve.

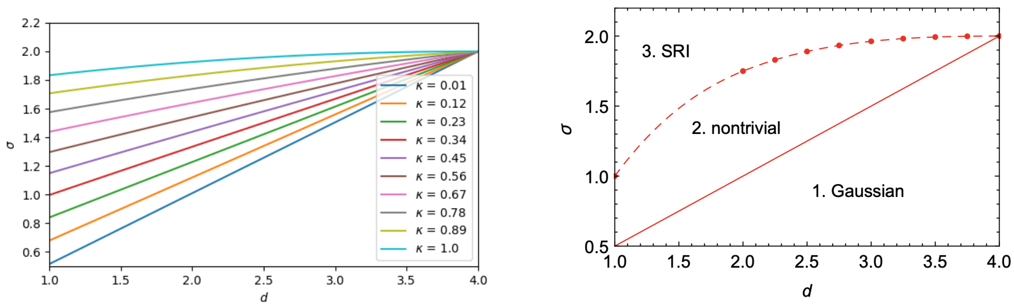


Figure 3: Left: The trajectories $\sigma(d)$ such that in the ε-expansion, $\gamma_\phi = 0$ in the case $N = 1$. Right: Repeat of figure 1 for comparison.

2.4 Expansion around another saddle-point

Before ending our discussion on the localized magnetic field, let's discuss how one might attempt to observe interesting dynamics close to $d = 3$ in the localized magnetic field in the long-range $O(N)$ model. This is apparently impossible from the previous analysis. Indeed, making the defect coupling marginally relevant forced us to choose d close to 4. However, we can describe the problem using a new action. Indeed, QFT can be seen as describing quantum corrections to classical field theory via the path integral, and typically one perturbs around a zero-valued vacuum:

$$\phi = \langle \phi \rangle + \delta\phi = \delta\phi \tag{2.33}$$

since in a theory without a defect, one can replace ϕ with $\phi + b$ for any constant b to effectively cancel vacuum expectation values. Therefore, strictly speaking, the path integral provides us with correlators of $\delta\phi$ – which coincides with ϕ.

However, in a defect theory, fields get a physical non-trivial one-point function. Hence, rather than expanding around $\langle \phi \rangle = 0$, one should expand around the DCFT ansatz

$$\phi(x) = \frac{a_\phi \sqrt{N_\phi}}{|x|_{d-1}^{\Delta_\phi}} + \delta\phi \tag{2.34}$$

This effectively amounts to studying the theory around another saddle point. Next, we use the fact that classically $\langle \phi^3 \rangle = \langle \phi \rangle^3$ and take the expectation value of th equation of motion for x in the bulk:

$$\mathcal{L}_\sigma \langle \phi \rangle = -\frac{\lambda_0}{3!} \langle \phi^3 \rangle \tag{2.35}$$

We can then solve for a_ϕ by plugging (2.34) into the equation of motion:

$$a_\phi^2 = -\frac{6\mathcal{N}_\sigma}{\lambda_0 N_\phi} \frac{\sqrt{\pi}\,\Gamma\left(\frac{d+\sigma-1}{2}\right)}{\Gamma\left(\frac{d+\sigma}{2}\right)} \frac{w^{(d-1)}_{d+\sigma-1} w^{(d-1)}_{\frac{d-\sigma}{2}}}{w^{(d-1)}_{\frac{d+\sigma}{2}}} = \frac{(1+\kappa)(N+8)}{8\kappa} + \mathcal{O}(\varepsilon) \tag{2.36}$$

where, as a consistency check, we've verified that to zeroth order in ε, we recover the right value for the one-point data in $d = 4 - \varepsilon$ close to the LRI fixed point. Incidentally, this method shows that once can alternatively renormalize

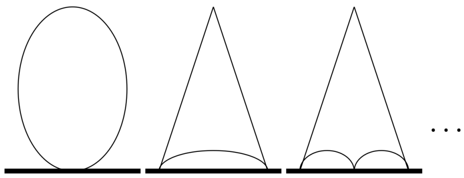


Figure 4: Diagrams involved in the one-point function of ϕ^2. The diagrams in this section come from [5].

the bulk and use the equations of motion to derive the fixed point defect coupling, rather than directly use the defect Feynman diagrams. Now that we have a_ϕ, we can derive the action for $\delta\phi$ around the new saddle-point[12]:

$$S = \frac{1}{2}\int d^d x \delta\phi_a(x)\mathcal{L}_\sigma \delta\phi_a(x) + \frac{\lambda_0}{4!}\int d^d x \left(\frac{2N_\phi a_\phi^2\left(\delta\phi_a(x)^2 + 2\delta\phi_1(x)^2\right)}{|x|_{d-1}^{2\Delta_\phi}} + \frac{4\sqrt{N_\phi}a_\phi\delta\phi_1(x)^3}{|x|_{d-1}^{\Delta_\phi}} + \left(\phi_a(x)^2\right)^2\right) \tag{2.37}$$

Hence, the propagator of $\delta\phi$ is shifted relative to that of ϕ, and quite surprisingly it seems that h is absent from this new theory. Nevertheless, constraining the solution to the bulk equation of motion to provide a solution emulating the presence of a defect seems to encode h_*. Incidentally, the $O(N)$ symmetry has been explicitly broken to $O(N-1)$ in the bulk. Studying Feynman diagrams in this new theory close to $d=3$, one can hope to obtain interesting dynamics, although this lies beyond the scope of the present report and we leave this for later work.

3 Quadratic surface defect in the long-range $O(N)$ model

Let's study the next simplest case of a defect in the long-range $O(N)$ model, namely the surface defect [3, 5] defined below:

$$S = \frac{\mathcal{N}_\sigma}{2}\int d^d x d^d y \frac{\phi_a(x)\phi_a(y)}{|x-y|^{d+\sigma}} + \frac{\lambda_0}{4!}\int d^d x \left(\phi_a(x)^2\right)^2 + h_0\int d^2 x \phi_1(x)^2 \tag{3.1}$$

3.1 ε-expansion

Consider the one-point function of ϕ_1^2. The relevant diagrams are given in figure 4. In momentum space, the renormalization of the one-point function of ϕ_1^2 is equivalent to that of $\langle\phi_1(k,m)\phi_1(-k,n)\rangle$ where k is the momentum along the defect and m,n are orthogonal to the defect. The associated diagrams form a geometric sequence (this argument is adapted from [5]).

$$\begin{cases} A_0 = -2h_0\frac{N}{(k^2+m^2)^{\frac{\sigma}{2}}(k^2+n^2)^{\frac{\sigma}{2}}} \\ A_{n+1} = tA_n \end{cases} \tag{3.2}$$

with

$$t\left(k\right) = -2h_0\int\frac{d^{d-2}p}{(2\pi)^{d-2}}\frac{1}{\left(k^2+p^2\right)^{\frac{\sigma}{2}}} = -\frac{2h_0 k^{d-2-\sigma}\Gamma\left(1-\frac{d-\sigma}{2}\right)}{(4\pi)^{\frac{d-2}{2}}\Gamma\left(\sigma\right)} = -\frac{2h}{(1+\kappa)\pi\varepsilon} + \mathcal{O}(1) \tag{3.3}$$

Hence we can take:

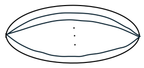
$$h_0 = \mu^{\frac{1+\kappa}{2}\varepsilon}h\left(1 + \frac{2h}{(1+\kappa)\pi\varepsilon} + \left(\frac{2h}{(1+\kappa)\pi\varepsilon}\right)^2 + \ldots\right) = \frac{\mu^{\frac{1+\kappa}{2}\varepsilon}h}{1-\frac{2h}{(1+\kappa)\pi\varepsilon}} \tag{3.4}$$

This yields the following β-function, which is exact (i.e. non-perturbative in the coupling):

$$\beta_h = -\frac{1+\kappa}{2}\varepsilon h + \frac{h^2}{\pi} \tag{3.5}$$

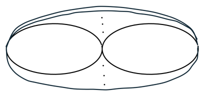
and one can then easily check that the non-trivial fixed point is $h_* = \frac{\pi(1+\kappa)}{2}\varepsilon \ll 1$. Next, let's consider the interacting theory. Before renormalizing the coupling, we need to figure out the wavefunction renormalization of ϕ^2. Contrary to ϕ, ϕ^n for $n>1$ renormalizes at one-loop via the following diagrams:

$Nn!$



$$= \frac{Nn!N_\phi^n}{|x|^{2n\Delta_\phi}} \tag{3.6}$$

[12]Recall that a solution to the equation of motion satisfies $\delta S/\delta\phi_a = 0$, hence we can discard terms linear in $\delta\phi_a$ in the new action.

$$\frac{N(N+2)n!n(n-1)}{12}$$

$$= -\frac{N(N+2)n!n(n-1)N_\phi^{n+2}\lambda_0}{12}\frac{\left[w^{(d)}_{4\Delta_\phi}\right]^2}{w^{(d)}_{8\Delta_\phi-d}}\frac{1}{|x|^{(8+2n-4)\Delta_\phi-d}} \tag{3.7}$$

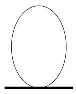
The second diagram diverges, and introducing Z_n such that $\phi^n = Z_n[\phi^n]$ [13] leads to:

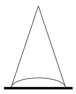
$$Z_n = 1 - \frac{(N+2)n(n-1)}{6\kappa\varepsilon}\frac{\lambda}{(4\pi)^2} + \mathcal{O}\left(\frac{\lambda^2}{(4\pi)^4}\right) \tag{3.8}$$

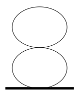
Evaluating the Feynman diagrams in the renormalized theory gives us the following normalization factor for the bulk two-point function of ϕ^n:

$$N_{\phi^n} = \frac{Nn!}{16\pi^4}\left[1 + \left(\gamma\kappa + (1-\kappa)\log 2 + \log\pi - \frac{n(n-1)}{2(N+8)}(N+2)(1+\gamma(2\kappa-1) + 2(1-\kappa)\log 2 + \log\pi)\right)\varepsilon + \mathcal{O}\left(\varepsilon^2\right)\right] \tag{3.9}$$

Next, the coupling h is renormalized via the following diagrams to one-loop order. This time, $[h] = (1+\kappa)\varepsilon/2$.

$$2N$$

$$= -2h_0 N_\phi^2 N\frac{\pi}{2\Delta_\phi - 1}\frac{1}{|x|^{4\Delta_\phi-2}_{d-2}} \tag{3.10}$$

$$4N$$

$$= 4N_\phi^3 N h_0^2 \frac{\pi^3\Gamma\left(2\Delta_\phi-1\right)^2\Gamma\left(3\Delta_\phi-2\right)}{\Gamma(\Delta_\phi)^3\Gamma\left(4\Delta_\phi-2\right)\sin(\pi\Delta_\phi)}\frac{1}{|x|^{6\Delta_\phi-4}_{d-2}} \tag{3.11}$$

$$\frac{N(N+2)}{3}$$

$$= \frac{h_0\lambda_0 N_\phi^4 N(N+2)}{3}\frac{\left[w^{(d)}_{4\Delta_\phi}\right]^2}{w^{(d)}_{8\Delta_\phi-d}}\frac{\pi}{4\Delta_\phi - 1 - \frac{d}{2}}\frac{1}{|x|^{8\Delta_\phi-d-2}_{d-2}} \tag{3.12}$$

This yields the following renormalization for h at two-loops (recall that we need to divide the sum of the above diagrams by Z_2):

$$\begin{cases} h_0 = \mu^{\frac{1+\kappa}{2}\varepsilon}h\left(\frac{1}{1-\frac{2h}{(1+\kappa)\pi\varepsilon}} + \frac{(N+2)\lambda}{48\pi^2\kappa\varepsilon}\right) + \mathcal{O}\left(h^3, h^2\lambda, h\lambda^2, \lambda^3\right) \\ \beta_h = -\frac{1+\kappa}{2}\varepsilon h + \frac{h^2}{\pi} + \frac{N+2}{48\pi^2}h\lambda \\ h_* = \frac{1+\kappa}{2}\pi\varepsilon - \frac{N+2}{48\pi}\lambda \end{cases} \tag{3.13}$$

At the bulk fixed-point:

$$h_* = \frac{N\left(1-\kappa\right) + 8 + 4\kappa}{2\left(N+8\right)}\pi\varepsilon + \mathcal{O}\left(\varepsilon^2\right) \tag{3.14}$$

Contrary to the case of the localized magnetic field, there is no divergence at $\kappa = 0$. Moreover, the fixed point is perturbative in ε, rather than finite as previously.

3.2 Some defect observables

Scaling dimension. As before, a derivative of the β-function for h gives us access to the following defect scaling dimension at the fixed point:

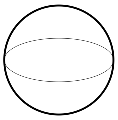
$$\Delta_{\hat{\phi}^2} = 2 + \frac{8 + 4\kappa + N(1-\kappa)}{2(N+8)}\varepsilon + \mathcal{O}(\varepsilon^2) \tag{3.15}$$

One point data. Using the Feynman diagrams for $\langle[\phi_1^2]\rangle$ and including the renormalization factors, we arrive at:

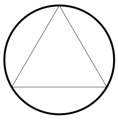
$$a_{\phi^2} = -\frac{\sqrt{N}(4\kappa + N(1-\kappa) + 8)}{4\sqrt{2}(N+8)}\varepsilon + \mathcal{O}(\varepsilon^2) \tag{3.16}$$

[13] While we don't use it here, we can easily derive the anomalous dimension of ϕ^n: $\gamma_n(\lambda_*) = \beta_\lambda \frac{\partial \log Z_n}{\partial \lambda} = \frac{3n(n-1)(N+2)}{6(N+8)}\kappa\varepsilon + \mathcal{O}\left(\varepsilon^2\right)$.

Defect free energy. A last observable that is worth deriving is the defect free energy $\mathcal{F}$. It is given by the following diagrams to first order in λ:

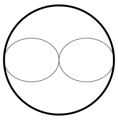


$$= h_0^2 N_\sigma^2 N \oint \frac{d^2x d^2y}{|x-y|^{4\Delta_\phi}} = -h_0^2 N_\sigma^2 N \frac{(2R)^{4(1-\Delta_\phi)}\pi^2}{2\Delta-1} \tag{3.17}$$



$$\begin{aligned} &= -h_0^3 N_\sigma^3 \frac{4N}{3} \oint \frac{d^2x d^2y d^2z}{|x-y|^{2\Delta_\phi}|y-z|^{2\Delta_\phi}|z-x|^{2\Delta_\phi}} \\ &= -h_0^3 N_\sigma^3 \frac{4N}{3} R^{6(1-\Delta_\phi)} \frac{8\pi^{\frac{9}{2}}\Gamma\left(2-3\Delta_\phi\right)}{\Gamma\left(\frac{3}{2}-\Delta_\phi\right)^3} \end{aligned} \tag{3.18}$$

where the integral above can be evaluated using a more general integral, defined for example via equations (14) and (17) in [34] and first used in [35].



$$= -\lambda_0 h_0^2 N_\sigma^4 \frac{N(N+2)}{6} \frac{\left[w_{4\Delta_\phi}^{(d)}\right]^2}{w_{8\Delta_\phi-d}^{(d)}} \frac{(2R)^{d+4-8\Delta_\phi}\pi^2}{\frac{d}{2}+1-4\Delta_\phi} \tag{3.19}$$

All of this yields the following logarithmic term in the free energy, which is the only scheme-independent one [5]:

$$\mathcal{F} = \left[\frac{N(1+\kappa)\varepsilon h^2}{16\pi^2} - \frac{Nh^3}{12\pi^3} - \frac{N(N+2)\lambda h^2}{384\pi^4}\right] \log\left(\mu R\right) + \dots \tag{3.20}$$

which is very close to the local case apart from the κ dependence in the first term [5]. Conventionally, one writes $\mathcal{F} = -b/3\log(\mu R) + \dots$. Hence, at the defect fixed point:

$$b_{\mathrm{IR}} = -\frac{(8+4\kappa+N(1-\kappa))^3}{64(N+8)^3}\varepsilon^3 + \mathcal{O}\left(\varepsilon^4\right) < b_{\mathrm{UV}} = 0 \tag{3.21}$$

This is consistent with the F-theorem.

4 Interaction between generalized free fields on a defect

4.1 Setup

In order to build more non-trivial defects in the long-range $O(N)$ model, the next step is to add additional fields (we now consider $N = 1$ for simplicity). For example, one might add an additional GFF ψ on the defect and consider the following class of actions:

$$S = \int_d \left[\frac{1}{2}\phi\mathcal{L}_{d-2\Delta_\phi}\phi + \frac{\lambda}{4!}\phi^4\right] + \frac{1}{2}\int_p \psi\mathcal{L}_{p-2\Delta_\psi}\psi + \frac{g}{2}\int_p \phi^a\psi^b \tag{4.1}$$

To reach the LRI fixed point and trigger an RG flow on the defect, we tune Δ_ϕ to $d/4$ and require:

$$\Delta_\psi = \frac{p-a\Delta_\phi}{b} = \frac{4p-ad}{4b} \tag{4.2}$$

Let's now impose unitarity. This can be written:

$$\Delta_\psi \geq \max\left(0, \frac{p-2}{2}\right) \tag{4.3}$$

in other words, the two following inequalities must hold

$$\begin{cases} ad \leq 4p \\ ad \leq 4p - 2b(p-2) \end{cases} \tag{4.4}$$

Let's look at each case depending on the value of p.

- $\boldsymbol{p=1}$: If $d=3$ or $d=4$, $a=1$. Any value of b can be used here. Moreover, it's worth noting that if $ad=4$, then $d=4$[14] and the additional operator on the defect becomes constant i.e. the identity (see equation (4.3)).
- $\boldsymbol{p=2}$: If $d=3$ or $d=4$, $a \leq 2$. If the upper bound is saturated, $ad=8$ and we take $d=4$ and $a=2$. Again, there is no constraint on b.

[14] Recall that we ignore $d=2$ cases in this work.

- $\boldsymbol{p = 3}$: We only look at $d = 4 > p$. Then $b \leq 6 - 2a$. If $a = 1$, $b \leq 4$, if $a = 2$, $b \leq 2$.

Let's focus on $p = 1$ or 2. It turns out that saturating the unitarity bound in $d = 4$ leads to $\Delta_\psi \to 0$ and ψ behaves as the identity. Hence, the defect interaction looks a lot like $h_0\phi$ or $h_0\phi^2$ i.e. the defects we studied previously, and it appears that fixing $a = 1$ or $a = 2$ for $p = 1$ or $p = 2$ respectively, and taking $d = 4 - \varepsilon$ leads to a possible matchup with the localized magnetic field or with the surface defect. If we introduce another scaling coefficient ν such that $[g] = \nu\varepsilon$ for example, then $\Delta_\psi \propto (p(1+\kappa) - 4\nu)$ and we can choose the scaling coefficients such that $\Delta_\psi = 0$ exactly. We'll examine the existence of a possible matchup in this case, simplifying our study to the case of a free bulk.

4.2 Matchup with line and surface defects

In this section, we briefly argue that in some cases where the defect GFF can be integrated out, there is no matchup with the localized magnetic field and the surface defect when $d = 4 - \varepsilon$. Let's look at the $b = 1$ case first. Integrating out the defect GFF and taking $d \to 4$ in the $a = p$ case yields to lowest nontrivial order (see appendix G.1) :

$$S_{\text{eff}}[\phi] = \frac{1}{2}\int d^dx\phi\mathcal{L}_\sigma\phi - \left(g_0\int d^px\phi^p\right)^2 \quad \text{[15]} \tag{4.5}$$

since $\Delta_\psi \to 0$. This action clearly can't be mapped to the localized magnetic field, nor to the surface defect by a simple redefinition of g_0. As for $b = 2, a = p$, the effective action is (see appendix G.2)

$$S_{\text{eff}}[\phi] = \frac{1}{2}\int d^dx\phi\mathcal{L}_\sigma\phi + \sum_{j=1}^{+\infty}\frac{(-1)^{j+1}}{2j}g_0^j\left(\int d^px_i\phi(x_i)^a\right)^j = \frac{1}{2}\int d^dx\phi\mathcal{L}_\sigma\phi + \frac{1}{2}\ln\left(1 + g_0\int d^px\phi(x)^a\right) \tag{4.6}$$

This action seems a lot closer to the localized magnetic field and the surface defect. As we shall soon see, g renormalizes in a few of the interesting cases, and the fixed point is infinitesimal. Hence, expanding the logarithm leads to:

$$S_{\text{eff}} = \frac{1}{2}\int d^dx\phi\mathcal{L}_\sigma\phi + \frac{g_0}{2}\int d^px\phi(x)^a + \ldots \tag{4.7}$$

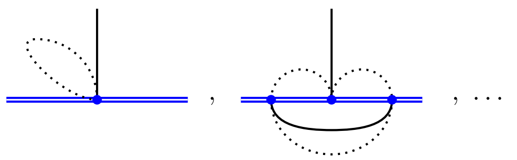
Therefore, at the lowest orders, the theory does resemble the line or surface defects we studied. Notwithstanding, their behaviors diverge from one another as we include higher loop orders. One observable which exemplifies this is the g-function for the $a = p = 1$ theory, which is readily accessible from the effective theory:

$$g = \sum_{k=0}^{\infty}\frac{(4k)!}{(2k)!k!}\left[g_0^2 2^{-3-2\Delta_\phi}R^{2-2\Delta_\phi}\pi\frac{\Gamma\left(\frac{1}{2} - \Delta_\phi\right)\sqrt{\pi}}{\Gamma(1-\Delta_\phi)}\right]^k \tag{4.8}$$

The above expression cannot directly be mapped to (2.23) by a redefinition of g_0. This is essentially because of the coefficients $(4k)!/(2k)!$ making it depart from the series of an exponential. Incidentally, these coefficients also translate the different combinatorics in the two theories on the line. Indeed, when $\Delta_\psi \to 0$, some diagrams which were previously distinct become the same, and contribute non-trivially to the combinatorics.

4.3 General statements

In the remainder of this report, we drop bulk interactions, since this class of theories already behaves non-trivially in this case. Moreover, lower order behavior for an interacting bulk can be extrapolated from this scenario. Before we look at specific values of (a, b), we establish some useful results which hold for all RG flows studied in this section. For a given (a, b), we conduct an ε-expansion such that $[g] = \varepsilon$, which implies $\Delta_\psi = (p - a\Delta_\phi - \varepsilon)/b$. In some of these theories, for example $(a, b) = (1, 2)$, one-point functions involve tadpoles and can't be used as in the case of the localized magnetic field to renormalize the coupling. Indeed, this would involve diagrams such as



(4.9)

both of which include tadpoles, and one can check that higher order diagrams also include such singular behavior. Since the theory is scaleless, these can be thrown away, and one can set $a_\phi = 0$; this does not help with the derivation of a β-function.

[15] We ignore the normalization factor in front of g_0, since the coupling can be redefined to absorb it as we attempt to map to the localized magnetic field or the surface defect.

Instead, one can look at the defect two-point function of a special composite operator, $O(x) = \phi^{a-1}(x)\psi^2(x)$, involved in the equation of motion for ϕ. Indeed:

$$\mathcal{L}_{d-2\Delta_\phi}\phi(x) = -\frac{g_0}{2}a\phi^{a-1}\psi^b\delta^{(d-p)}(x) \tag{4.10}$$

However, as previously mentioned, nonlocal theories do not renormalize, and since $d < 4$ we are not placing ourselves close to the local free theory limit. Hence, it follows that $Z_g Z_O = 1$. One can then derive renormalization of the coupling from the wavefunction renormalization of O. Furthermore, one can invert the above equation of motion to derive the one-point function of ϕ^2, given the two-point function of O. It turns out that we can already say a lot about it without calculating Feynman diagrams in the various specific cases. Indeed, given a renormalization factor of the following form:

$$Z_O = 1 - \frac{\alpha g^n}{\varepsilon} + \mathcal{O}\left(g^{n+1}\right) \tag{4.11}$$

where n is some integer and α is some real number, the theory admits a β-function satisfying

$$\beta_g = -\varepsilon g + \alpha n g^{n+1} + \mathcal{O}\left(g^{n+1}\right) \tag{4.12}$$

and such a β-function leads to a non-trivial fixed point $g_*^n = \varepsilon/(\alpha n) + \mathcal{O}\left(\varepsilon^2\right)$. At this fixed point, O gains an anomalous dimension:

$$\gamma_O = \beta_g \frac{\partial \ln Z_O}{\partial g} = \varepsilon + \mathcal{O}\left(\varepsilon^2\right) \tag{4.13}$$

The marginal relevance of the interaction implies $a\Delta_\phi + b\Delta_\psi = p - \varepsilon$. Hence $\Delta_O = (a-1)\Delta_\phi + b\Delta_\psi + \gamma_O = p - \Delta_\phi$ and the above operator's dimension is always protected throughout the RG flow. Next, we can invert the previous equation of motion at the fixed point for x in the bulk:

$$\phi\left(x\right) = -a\mathcal{N}_{2\Delta_\phi - d}\frac{g_0}{2}\int d^d y \frac{O(y)\delta^{(d-p)}(y)}{|x-y|^{2\Delta_\phi}} = -a\mathcal{N}_{2\Delta_\phi - d}\frac{g_0}{2}\int d^p y \frac{O(y)}{\left(|x|^2_{d-p} + |y|^2_p\right)^{\Delta_\phi}} \tag{4.14}$$

This allows us to easily deduce the one-point function of ϕ^2.

$$\begin{aligned}\langle\phi^2(x)\rangle &= g_0^2\frac{a^2\mathcal{N}^2_{2\Delta_\phi-d}}{4}N_O\int\frac{d^p y_1 d^p y_2}{\left(|x|^2_{d-p}+|y_1|^2_p\right)^{\Delta_\phi}|y_1-y_2|^{2(p-\Delta_\phi)}\left(|x|^2_{d-p}+|y_2|^2_p\right)^{\Delta_\phi}} \\ &= g_0^2\frac{a^2\mathcal{N}^2_{2\Delta_\phi-d}}{4}\frac{\pi^p\Gamma\left(\frac{p}{2}\right)\Gamma\left(\Delta_\phi-\frac{p}{2}\right)}{\Gamma\left(\Delta_\phi\right)(p-1)!}\frac{N_O}{|x|^{2\Delta_\phi}_{d-p}} = \frac{a_{\phi^2}N_\phi}{|x|^{2\Delta_\phi}_{d-p}}\end{aligned} \tag{4.15}$$

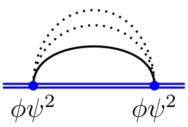
where typically $N_O = (a-1)!b!N_\phi^{a-1}N_\psi^b + \mathcal{O}(\varepsilon^2)$. Another observable that can generally be examined is the two-point function $\langle\phi(x)O(y)\rangle_D$ on the defect. Note that $\Delta_O - \Delta_\phi = 2\Delta_\phi - p \leq 2 - p \leq 1$. Hence one can expect in many cases that O not be a descendant of ϕ, and hence a non-trivial test of conformality might be to verify that $\langle\phi(x)O(y)\rangle_D = 0$. This is indeed the case:

$$\langle\phi(x)O(y)\rangle_D = -a\mathcal{N}_{2\Delta_\phi-d}\frac{g_0}{2}\int\frac{d^p z}{|x-z|^{2\Delta_\phi}|z-y|^{2(p-\Delta_\phi)}} \propto \delta^{(p)}(x-y) = 0 \tag{4.16}$$

Let's look at a few simple cases before concluding this work.

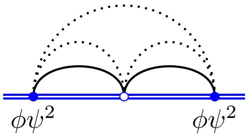
4.4 $(a,b) = (2,2)$

The simplest theory as far as renormalization is concerned is $(a,b) = (2,2)$. In this case $O = \phi\psi^2$, let's calculate its two-point function. At tree-level this is given by the following diagram (dotted lines stand for free propagators of ψ)

2 

$$= \frac{2N_\phi N_\psi^2}{|x|^{2\Delta_\phi+4\Delta_\psi}} \tag{4.17}$$

The only contribution at one-loop comes from:

2 

$$= -4g_0 N_\phi^2 N_\psi^3 \frac{\left[w^{(p)}_{2\Delta_\phi+2\Delta_\psi}\right]^2}{w^{(p)}_{4\Delta_\phi+4\Delta_\psi-p}}\frac{1}{|x|^{4\Delta_\phi+6\Delta_\psi-p}} \tag{4.18}$$

This diagram diverges, and its pole can be canceled by introducing the following wavefunction renormalization:

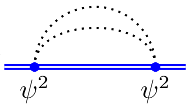
$$\begin{cases} Z_{\phi\psi^2} = 1 - \frac{\Gamma\left(\frac{p}{2}-\Delta_\phi\right)g}{2^{d-2}\pi^{\frac{d}{2}}\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{d}{2}-\Delta_\phi\right)\varepsilon} + \mathcal{O}\left(g^2\right) \\ \beta_g = -\varepsilon g + \frac{\Gamma\left(\frac{p}{2}-\Delta_\phi\right)}{2^{d-2}\pi^{\frac{d}{2}}\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{d}{2}-\Delta_\phi\right)}g^2 + \mathcal{O}\left(g^3\right) \\ g_* = \frac{2^{d-2}\pi^{\frac{d}{2}}\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{d}{2}-\Delta_\phi\right)}{\Gamma\left(\frac{p}{2}-\Delta_\phi\right)}\varepsilon + \mathcal{O}\left(\varepsilon^2\right) \end{cases} \tag{4.19}$$

Inverting the equation of motion as previously explained yields the following CFT datum at the fixed point:

$$a_{\phi^2} = \frac{\Gamma\left(\Delta_\phi - \frac{p}{2}\right)\Gamma\left(\frac{p}{2}\right)^3}{2\Gamma\left(\Delta_\phi\right)(p-1)!}\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right) \tag{4.20}$$

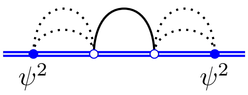
4.5 $(a,b) = (1,2)$

In this case, $O = \psi^2\psi^2$. At tree-level this is given by the following diagram (dotted lines stand for free propagators of ψ)

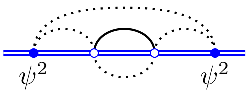
2 

$$= \frac{2N_\phi^2}{|x|^{4\Delta_\psi}} \tag{4.21}$$

At one-loop, the relevant diagrams are:

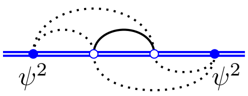
$\frac{2!2!2}{2^2}$ 

$$= 2N_\phi N_\psi^4 g_0^2 \frac{\left[w^{(p)}_{4\Delta_\psi}\right]^2 w^{(p)}_{2\Delta_\phi}}{w^{(p)}_{8\Delta_\psi+2\Delta_\phi-2}} \frac{1}{|x|^{8\Delta_\psi+2\Delta_\phi-2}} \tag{4.22}$$

$\frac{16}{2^2}$ 

$$= \frac{4N_\phi N_\psi^4 g_0^2\left[w^{(p)}_{2\Delta_\psi}\right]^2 w^{(p)}_{2\Delta_\phi+2\Delta_\psi}}{w^{(p)}_{6\Delta_\psi+2\Delta_\phi-2}} \frac{1}{|x|^{8\Delta_\psi+2\Delta_\phi-2}} \tag{4.23}$$

The third diagram's computation is much more involved, but one can extract its pole by studying how the diagram behaves when various interaction points coincide, and keep the cases where a logarithmic divergence appears. This only happens when the two interaction points are both at 0, or at x. Indeed, the diagram is given by:

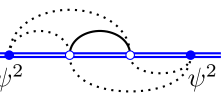
$\frac{2^4}{2^2}$ 

$$= 4N_\phi N_\psi^4 g_0^2 \int d^p y_1 d^p y_2 \frac{1}{|y_1|^{2\Delta_\psi}}\frac{1}{|y_2|^{2\Delta_\psi}}\frac{1}{|x-y_1|^{2\Delta_\psi}}\frac{1}{|x-y_2|^{2\Delta_\psi}}\frac{1}{|y_2-y_1|^{2\Delta_\phi}} \tag{4.24}$$

When $y_1, y_2 \to 0$, this diagram behaves as

$$\frac{1}{|x|^{4\Delta_\psi}}\int d^p y_1 d^p y_2 \frac{1}{|y_1|^{2\Delta_\psi}}\frac{1}{|y_2|^{2\Delta_\psi}}\frac{1}{|y_2-y_1|^{2\Delta_\phi}} = \left(\frac{w^{(p)}_{2\Delta_\phi}w^{(p)}_{2\Delta_\psi}}{w^{(p)}_{2\Delta_\phi+2\Delta_\psi-p}}\frac{\pi^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}\right)}\frac{1}{\varepsilon} + \mathcal{O}(1)\right)\frac{1}{|x|^{4\Delta_\psi}} \tag{4.25}$$

and the same behavior is obtained when $y_1, y_2 \to x$. Hence the following pole:

$\frac{2^4}{2^2}$ 

$$\supset 4N_\phi N_\psi^4 g_0^2\left(\frac{w^{(p)}_{2\Delta_\phi}w^{(p)}_{2\Delta_\psi}}{w^{(p)}_{2\Delta_\phi+2\Delta_\psi-p}}\frac{2\pi^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}\right)}\frac{1}{\varepsilon} + \mathcal{O}(1)\right)\frac{1}{|x|^{4\Delta_\psi}} \tag{4.26}$$

We can cancel this divergence by introducing wavefunction renormalization $\psi^2 = Z_{\psi^2}[\psi^2]$ with

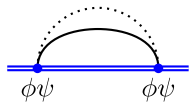
$$\begin{cases} Z_{\psi^2} = 1 + g^2\frac{\Gamma\left(\frac{p}{2}-\Delta_\phi\right)}{2^{d-1}\pi^{\frac{d}{2}}\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{d}{2}-\Delta_\phi\right)\varepsilon} + \mathcal{O}\left(g^4\right) \\ \beta_g = -\varepsilon g - \frac{\Gamma\left(\frac{p}{2}-\Delta_\phi\right)}{2^{d-2}\pi^{\frac{d}{2}}\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{d}{2}-\Delta_\phi\right)}g^3 + \mathcal{O}(g^4) \\ g_*^2 = -\frac{2^{d-2}\pi^{\frac{d}{2}}\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{d}{2}-\Delta_\phi\right)}{\Gamma\left(\frac{p}{2}-\Delta_\phi\right)}\varepsilon + \mathcal{O}(\varepsilon^2) \end{cases} \tag{4.27}$$

Using the previous general results and $N_{\psi^2} = 2N_\psi^2 + \mathcal{O}\left(\varepsilon^2\right)$, we arrive at the following DCFT data (normalized by N_ϕ):

$$a_{\phi^2} = -\frac{9\Gamma\left(\frac{p}{2}\right)^2\Gamma\left(\Delta_\phi - \frac{p}{2}\right)\Gamma\left(\frac{p-\Delta_\phi}{2}\right)^2}{32\Gamma\left(\frac{\Delta_\phi}{2}\right)^2\Gamma\left(\frac{p}{2}-\Delta_\phi\right)(p-1)!}\varepsilon + O\left(\varepsilon^2\right) \tag{4.28}$$

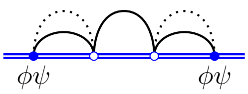
4.6 $(a,b) = (2,1)$

The last case we look at is $(a,b) = (2,1)$.



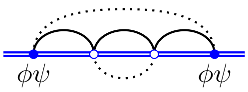
$$= \frac{N_\phi N_\psi}{|x|^{2(\Delta_\phi+\Delta_\psi)}} \tag{4.29}$$

$$\frac{2!2!2}{2^2}$$



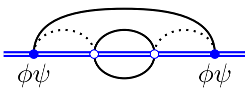
$$= 2g_0^2 N_\phi^3 N_\psi^2 \frac{\left[w^{(p)}_{2\Delta_\phi+2\Delta_\psi}\right]^2 w^{(p)}_{2\Delta_\phi}}{w^{(p)}_{6\Delta_\phi+4\Delta_\psi-2p}} \frac{1}{|x|^{6\Delta_\phi+4\Delta_\psi-2p}} \tag{4.30}$$

$$\frac{2!2!2}{2^2}$$



$$= 2g_0^2 N_\phi^3 N_\psi^2 \frac{\left[w^{(p)}_{2\Delta_\phi}\right]^2 w^{(p)}_{2\Delta_\phi+2\Delta_\psi}}{w^{(p)}_{6\Delta_\phi+2\Delta_\psi-2p}} \frac{1}{|x|^{6\Delta_\phi+4\Delta_\psi-2p}} \tag{4.31}$$

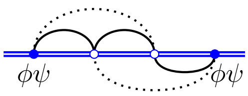
$$\frac{2!2}{2^2}$$



$$= g_0^2 N_\phi^3 N_\psi^2 \frac{\left[w^{(p)}_{2\Delta_\psi}\right]^2 w^{(p)}_{4\Delta_\phi}}{w^{(p)}_{4\Delta_\phi+4\Delta_\psi-2p}} \frac{1}{|x|^{6\Delta_\phi+4\Delta_\psi-2p}} \tag{4.32}$$

The last diagram at one-loop is difficult to evaluate, nevertheless one can extract its poles as described previously:

$$\frac{2^3}{2^2}$$



$$\supset \left(\frac{w^{(p)}_{2\Delta_\phi} w^{(p)}_{2\Delta_\psi}}{w^{(p)}_{2\Delta_\phi+2\Delta_\psi-p}} \frac{2\pi^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}\right)} \frac{1}{\varepsilon} + \mathcal{O}(1) \right) \frac{1}{|x|^{2\Delta_\phi+2\Delta_\psi}} \tag{4.33}$$

This yields the following renormalization factor, β-functions and fixed point:

$$\begin{cases} Z_{\phi\psi} = 1 + \frac{\Gamma\left(\frac{p}{2}-\Delta_\phi\right)^2 g^2}{2^{2d-p-1}\pi^{d-\frac{p}{2}}\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{d}{2}-\Delta_\phi\right)^2\varepsilon} + \mathcal{O}\left(g^4\right) \\ \beta_g = -\varepsilon g - \frac{\Gamma\left(\frac{p}{2}-\Delta_\phi\right)^2}{2^{2(d-1)-p}\pi^{d-\frac{p}{2}}\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{d}{2}-\Delta_\phi\right)^2} g^3 + \mathcal{O}\left(g^4\right) \\ g_*^2 = -\frac{2^{2(d-1)-p}\pi^{d-\frac{p}{2}}\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{d}{2}-\Delta_\phi\right)^2}{\Gamma\left(\frac{p}{2}-\Delta_\phi\right)^2}\varepsilon + \mathcal{O}(\varepsilon^2) \end{cases} \tag{4.34}$$

These allow us to derive the following CFT datum:

$$a_{\phi^2} = -\frac{9\Gamma\left(\Delta_\phi\right)\Gamma\left(\Delta_\phi-\frac{p}{2}\right)\Gamma\left(\frac{p}{2}\right)^2\Gamma\left(p-2\Delta_\phi\right)}{16\Gamma\left(2\Delta_\phi-\frac{p}{2}\right)\Gamma\left(\frac{p}{2}-\Delta_\phi\right)^2(p-1)!}\varepsilon + \mathcal{O}\left(\varepsilon^2\right) \tag{4.35}$$

5 Conclusions and outlook

In this work, we built non-trivial defects in nonlocal $O(N)$-invariant theories, such as the long-range $O(N)$ model. We generalized results regarding the localized magnetic field and the surface defect in $d = 4-\varepsilon$ dimensions to the case of the long-range $O(N)$ model. We showed that these theories renormalize, and we calculated some basic observables – mainly one-point data, the g-function and the free energy. In order to have a theory which renormalizes close to $d = 3$, we introduced a new defect, involving a new GFF interacting with the first one. We only studied this setup in a free bulk; however, further work will be conducted to introduce this defect to the long-range $O(N)$-model.

Following our study of the localized magnetic field and of the surface defect, one can also look at volume defects with an interaction going like ϕ^3 in $d = 4-\varepsilon$. This has already been done in the local case, and the generalization should be well within reach [36]. In order to study the nonlocal $O(N)$ family, one can also conduct a so-called *large N expansion*. This is an alternative to the ε-expansion, and both schemes can be used in combination to produce better predictions (see [37] for a review of large N expansions in QFT). Further work is also needed to elucidate the expansion around another saddle-point, introduced in our study of the localized magnetic field: for example, calculating the propagator in the new theory and computing some Feynman diagrams. Lastly, one could use the technology of the analytic bootstrap to further compute some observables to higher orders in ε, such as the defect two-point function, as has been done in the local case in [38].

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A Reminder: Feynman and Schwinger parameters

First attempts at computing Feynman diagrams usually involve recasting integrals over position or momentum into integrals over parameters x_i or t such that:

$$\frac{1}{A_1^{\alpha_1}\dots A_n^{\alpha_n}} = \frac{\Gamma(\alpha_1+\dots+\alpha_n)}{\Gamma(\alpha_1)\dots\Gamma(\alpha_n)}\int_0^1 dx_1\dots\int_0^1 dx_n \frac{\delta(1-x_1-\dots-x_n)x_1^{\alpha_1-1}\dots x_n^{\alpha_n-1}}{(x_1A_1+\dots x_nA_n)^{\alpha_1+\dots+\alpha_n}} \tag{A.1}$$

$$\frac{1}{|p|^\sigma} = \frac{1}{\Gamma\left(\frac{\sigma}{2}\right)}\int_0^{+\infty} t^{\frac{\sigma}{2}-1}e^{-t|p|^2}dt \tag{A.2}$$

where A_i are scalar functions of position or momentum. We use both of these formulations in derivations of integral identities.

B Propagator for the LRI

As a prerequisite to using Wick's theorem to calculate correlators we need to calculate the propagator of the GFF theory. It is given by the Green's function of the fractional Laplacian, which can be solved for using the Fourier transform:

$$G(x_1-x_2) = \int \frac{d^dp}{(2\pi)^d}\frac{e^{ip\cdot(x_1-x_2)}}{|p|^\sigma} \tag{B.1}$$

If it is consistent with CFT, this quantity must go like $|x-y|^{-2\Delta_\phi}$. To show that this holds, we use the following identity (easily derived using a change of variables in the definition of the Γ-function):

$$\frac{1}{|p|^\sigma} = \frac{1}{\Gamma\left(\frac{\sigma}{2}\right)}\int_0^{+\infty} t^{\frac{\sigma}{2}-1}e^{-t|p|^2}dt \tag{B.2}$$

Next, we plug this into equation (B.1). This implies (integrals can be swapped using Fubini's theorem):

$$G(x) = \frac{1}{\Gamma\left(\frac{\sigma}{2}\right)}\int_0^{+\infty} dt t^{\frac{\sigma}{2}-1}\int \frac{d^dp}{(2\pi)^d}e^{-t|p|^2+ipx} \tag{B.3}$$

The momentum space integral is a well-known Gaussian integral, whose evaluation yields:

$$G(x) = \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{\sigma}{2}\right)}\int_0^{+\infty} dte^{-\frac{|x|^2}{4t}}t^{\frac{\sigma-d}{2}-1} \tag{B.4}$$

Next, the change of variables $u=|x|^2/(4t)$ turns the remaining integral into a Γ-function, and we obtain the desired result:

$$G(x) = \frac{2^{d-\sigma}\Gamma\left(\frac{d-\sigma}{2}\right)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{\sigma}{2}\right)}\frac{1}{|x|^{d-\sigma}} \tag{B.5}$$

Hence the propagator satisfies the conformality constraints, since $\Delta_\phi = (d-\sigma)/2$ is the scaling dimension of the field ϕ. Incidentally, since all correlation functions are given by sums of products of correlation functions, this computation shows that the free Gaussian theory is conformally invariant.

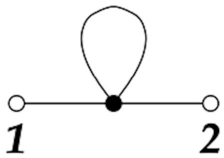


Figure 5: Tadpole appearing at one-loop order in the expansion of the two-point function in ϕ^4 theory.

C Useful Integrals

C.1 Tadpole integral

In this section, we derive a very standard integral in quantum field theory, but we include it nonetheless for the sake of completeness. This integral is called a tadpole integral, and a special case of it appears in expressions involving propagators evaluated at zero spacing i.e. "$G(0)$". It is defined – typically in a massive theory – by:

$$T_\alpha = \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2+m^2)^\alpha} \tag{C.1}$$

Going to hyperspherical coordinates, the integrand being a radial function of the vector p, we get:

$$T_\alpha = \frac{S_{d-1}}{(2\pi)^d} \int_0^{+\infty} \frac{p^{d-1}dp}{(p^2+m^2)^\alpha} = \frac{S_{d-1}}{(2\pi)^d} m^{-2\alpha} \int_0^{+\infty} p^{d-1} \left(1+\left(\frac{p}{m}\right)^2\right)^{-\alpha} dp \tag{C.2}$$

and then we perform a first change of variables $p = m\sqrt{y}$, arriving at:

$$T_\alpha = \frac{S_{d-1}}{2(2\pi)^d} m^{d-2\alpha} \int_0^{+\infty} y^{\frac{d}{2}-1}(1+t)^{-\alpha} dy \tag{C.3}$$

A second change of variables $t = 1/(1+y)$ allows us to extract an Euler B-function.

$$T_\alpha = \frac{S_{d-1} m^{d-2\alpha}}{2(2\pi)^d} B\left(\frac{d}{2}, \alpha - \frac{d}{2}\right) \tag{C.4}$$

Using the exact value of the unit $(d-1)$-sphere, which can incidentally be derived by looking at a d-dimensional Gaussian integral and going to spherical coordinates, we arrive at the final expression:

$$T_\alpha = \frac{(m^2)^{\frac{d}{2}-\alpha}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(\alpha - \frac{d}{2}\right)}{\Gamma(\alpha)} \tag{C.5}$$

Incidentally, for massless theories, this formula loosely justifies "throwing tadpoles away".

C.2 Loop integration

Integral over a bulk vertex. The following identity is often used in the calculation of Feynman diagrams in a CFT:

$$I(x) := \int \frac{d^d y}{|x-y|^A |y|^B} = \frac{w_A^{(d)} w_B^{(d)}}{w_{A+B-d}^{(d)}} \frac{1}{|x|^{A+B-d}} \tag{C.6}$$

where $w_A^{(d)} = (4\pi)^{d/2} 2^{-A} \Gamma\left(\frac{d-A}{2}\right)/\Gamma\left(\frac{A}{2}\right)$. The main idea is to calculate it in momentum space, and then perform an inverse Fourier transform back to position space. The Fourier transform of I obeys

$$\hat{I}(p) := \int \frac{d^d x d^d y e^{-ipx}}{|x-y|^A |y|^B} = \int \frac{d^d x d^d y e^{-ip(x+y)}}{|x|^A |y|^B} = (2\pi)^{2d} G_A(p) G_B(p) \tag{C.7}$$

where we've applied a translation by $+y$ to x, and $G_A(p) = K_A |p|^{A-d}$ is the Gaussian propagator for $\sigma = A$. Transforming back to position space yields

$$I(x) = (2\pi)^{2d} K_A K_B \int \frac{d^d p}{(2\pi)^d} \frac{e^{ipx}}{|p|^{2d-A-B}} = (2\pi)^{2d} K_A K_B K_{2d-A-B} \frac{1}{|x|^{A+B-d}} \tag{C.8}$$

Using the values of the K coefficients derived in the main text leads to the desired result.

Integral over a defect vertex.

$$\begin{aligned}\int \frac{d^p y}{|x-y|^\alpha} &= \int \frac{d^p y}{\left(|x|_{d-p}^2 + (x_{d-p+1}-y_{d-p+1})^2 + \cdots + (x_d - y_d)^2\right)^{\frac{\alpha}{2}}} \\ &= \frac{1}{|x|_{d-p}^{\alpha-p}} \int \frac{dy_{d-p-1}\ldots dy_d}{\left(1+y_{d-p+1}^2+\ldots+y_d^2\right)^{\frac{\alpha}{2}}} \\ &= S_p \int \frac{r^{p-1}dr}{(1+r^2)^{\frac{\alpha}{2}}} \\ &= \frac{S_p}{2}\int u^{\frac{p}{2}-1}(1+u)^{-\frac{\alpha}{2}}\,du \\ &= \frac{S_p}{2}\int_0^1 (1-y)^{\frac{p}{2}-1} y^{\frac{\alpha}{2}-\frac{p}{2}-1} du \\ &= \frac{S_p}{2}\frac{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{\alpha-p}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}\frac{1}{|x|_{d-p}^{\alpha-p}} \\ &= \frac{\pi^{\frac{p}{2}}\Gamma\left(\frac{\alpha-p}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}\frac{1}{|x|_{d-p}^{\alpha-p}}\end{aligned} \tag{C.9}$$

Integral over a defect vertex with bulk propagator.

$$\begin{aligned}\int \frac{d^d y}{|x-y|^\alpha\,|y|_p^\beta} &= \int \frac{d^p y}{|y|_p^\beta}\int \frac{dy_{p+1}\ldots dy_d}{\left||x-y|_p^2+y_{p+1}^2+\ldots.+y_d^2\right|^{\frac{\alpha}{2}}} \\ &= \int \frac{d^p y}{|y|_p^\beta\,|x-y|_p^{\alpha-(d-p)}}\int \frac{dy_{p+1}\ldots dy_d}{\left|1+y_{p+1}^2+\ldots.+y_d^2\right|^{\frac{\alpha}{2}}} \\ &= \frac{\pi^{\frac{d-p}{2}}\Gamma\left(\frac{\alpha-d+p}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}\int \frac{d^p y}{|y|_p^\beta\,|x-y|_p^{\alpha-(d-p)}} \\ &= \frac{\pi^{\frac{d-p}{2}}\Gamma\left(\frac{\alpha-d+p}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}\frac{w_{\alpha-d+p}^{(p)}w_\beta^{(p)}}{w_{\alpha+\beta-d}^{(p)}}\frac{1}{|x|_p^{\alpha+\beta-d}}\end{aligned} \tag{C.10}$$

Hard integral There is an integral which is useful in the renormalization of the surface defect, also used in [5]. The author uses it and cites a classic Feynman integration reference [39]. However, since this derivation is particularly hard and its derivation is not easily found online, we reproduce it here. We use Schwinger parameters, denoted t_i, below. We also use the inversion formula for the Γ-function.

$$F = \int \frac{d^d k d^d l}{(k^2+m^2)^{\lambda_1}\left[(k+l)^2\right]^{\lambda_2}(l^2+m^2)^{\lambda_3}} = \int \frac{d^d k}{(k^2+m^2)^{\lambda_1}} I(k) \tag{C.11}$$

$$\begin{aligned}I(k) &= \frac{1}{\Gamma(\lambda_2)\Gamma(\lambda_3)}\int_0^{+\infty} dt_2 dt_3 t_2^{\lambda_2-1}t_3^{\lambda_3-1}\int d^d l e^{-t_2(k+l)^2-t_3\left(l^2+m^2\right)} \\ &= \frac{1}{\Gamma(\lambda_2)\Gamma(\lambda_3)}\int_0^{+\infty} dt_2 dt_3 t_2^{\lambda_2-1}t_3^{\lambda_3-1}e^{-t_2k^2-t_3m^2+\frac{t_2^2k^2}{t_2+t_3}}\int d^d l e^{-(t_2+t_3)l^2} \\ &= \frac{\pi^{\frac{d}{2}}}{\Gamma(\lambda_2)\Gamma(\lambda_3)}\int_0^{+\infty} dt_2 dt_3 t_2^{\lambda_2-1}t_3^{\lambda_3-1}(t_2+t_3)^{-\frac{d}{2}}e^{-t_2k^2-t_3m^2+\frac{t_2^2k^2}{t_2+t_3}}\end{aligned} \tag{C.12}$$

$$
\begin{aligned}
F &= \frac{\pi^{\frac{d}{2}}}{\Gamma(\lambda_2)\Gamma(\lambda_3)} \int_0^{+\infty} dt_2 dt_3 t_2^{\lambda_2-1} t_3^{\lambda_3-1} (t_2+t_3)^{-\frac{d}{2}} e^{-t_3 m^2} \int \frac{d^d k}{(k^2+m^2)^{\lambda_1}} e^{-\left(t_2-\frac{t_2^2}{t_2+t_3}\right)k^2} \\
&= \frac{\pi^{\frac{d}{2}}}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)} \int_0^{+\infty} dt_1 dt_2 dt_3 t_1^{\lambda_1-1} t_2^{\lambda_2-1} t_3^{\lambda_3-1} (t_2+t_3)^{-\frac{d}{2}} e^{-(t_1+t_3)m^2} \int d^d k e^{-\left(t_1+t_2-\frac{t_2^2}{t_2+t_3}\right)k^2} \\
&= \frac{\pi^{d}}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)} \int_0^{+\infty} dt_1 dt_2 dt_3 t_1^{\lambda_1-1} t_2^{\lambda_2-1} t_3^{\lambda_3-1} (t_1 t_2 + t_1 t_3 + t_2 t_3)^{-\frac{d}{2}} e^{-(t_1+t_3)m^2} \\
&= \frac{\pi^{d}}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)} \int_0^{+\infty} dt_1 dt_2 dt_3 t_1^{\lambda_1-\frac{d}{2}-1} t_2^{\lambda_2-1} t_3^{\lambda_3-\frac{d}{2}-1} \left(\frac{t_1+t_3}{t_1 t_3} t_2 + 1\right)^{-\frac{d}{2}} e^{-(t_1+t_3)m^2} \\
&= \frac{\pi^{d}}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)} \int_0^{+\infty} dt_1 dy dt_3 t_1^{\lambda_1+\lambda_2-\frac{d}{2}-1} t_3^{\lambda_3+\lambda_2-\frac{d}{2}-1} (t_1+t_3)^{-\lambda_2} e^{-(t_1+t_3)m^2} y^{\lambda_2-1} (y+1)^{-\frac{d}{2}} \\
&= \frac{\pi^{d}\Gamma\left(\frac{d}{2}-\lambda_2\right)}{\Gamma(\lambda_1)\Gamma(\lambda_3)\Gamma\left(\frac{d}{2}\right)} \int_0^{+\infty} dt_1 dt_3 t_1^{\lambda_1+\lambda_2-\frac{d}{2}-1} t_3^{\lambda_3+\lambda_2-\frac{d}{2}-1} (t_1+t_3)^{-\lambda_2} e^{-(t_1+t_3)m^2} \\
&= \frac{\pi^{d}\Gamma\left(\frac{d}{2}-\lambda_2\right)}{\Gamma(\lambda_1)\Gamma(\lambda_3)\Gamma\left(\frac{d}{2}\right)} \int_0^{+\infty} du dv v^{\lambda_1+\lambda_2-\frac{d}{2}-1} (u-v)^{\lambda_3+\lambda_2-\frac{d}{2}-1} u^{-\lambda_2} e^{-um^2} \\
&= (-1)^{\lambda_1+\lambda_2-\frac{d}{2}} \frac{\pi^{d}\Gamma\left(\frac{d}{2}-\lambda_2\right)}{\Gamma(\lambda_1)\Gamma(\lambda_3)\Gamma\left(\frac{d}{2}\right)} \frac{\Gamma(1+d-\lambda_1-2\lambda_2-\lambda_3)\Gamma\left(\lambda_1+\lambda_2-\frac{d}{2}\right)}{\Gamma\left(1+\frac{d}{2}-\lambda_3-\lambda_2\right)} \int_0^{+\infty} du u^{\lambda_1+\lambda_2+\lambda_3-d-1} e^{-um^2} \\
&= \frac{(-1)^{\lambda_1+\lambda_2-\frac{d}{2}}}{(m^2)^{\lambda_1+\lambda_2+\lambda_3-d}} \frac{\pi^{d}\Gamma\left(\frac{d}{2}-\lambda_2\right)}{\Gamma(\lambda_1)\Gamma(\lambda_3)\Gamma\left(\frac{d}{2}\right)} \frac{\Gamma(1+d-\lambda_1-2\lambda_2-\lambda_3)\Gamma\left(\lambda_1+\lambda_2-\frac{d}{2}\right)}{\Gamma\left(1+\frac{d}{2}-\lambda_3-\lambda_2\right)} \Gamma(\lambda_1+\lambda_2+\lambda_3-d) \\
&= \frac{\pi^{d}\Gamma\left(\lambda_1+\lambda_2-\frac{d}{2}\right)\Gamma\left(\lambda_2+\lambda_3-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-\lambda_2\right)\Gamma(\lambda_1+\lambda_2+\lambda_3-d)}{\Gamma(\lambda_1)\Gamma(\lambda_3)\Gamma(\lambda_1+2\lambda_2+\lambda_3-d)\Gamma\left(\frac{d}{2}\right)} \frac{1}{(m^2)^{\lambda_1+\lambda_2+\lambda_3-d}}
\end{aligned}
\tag{C.13}
$$

C.3 Integration on the circle

Integral over one angle

$$
\begin{aligned}
\oint_{z\in\mathcal{C}} \frac{dz}{|y-z|^{2\Delta}} &= \int_{-\pi}^{\pi} \frac{R d\theta}{\left(y_{d-1}^2 + y_d^2 + |y|_{d-2}^2 + R^2 - 2R\sqrt{y_{d-1}^2+y_d^2}\cos\theta\right)^{\Delta}} \\
&= \int_{-\pi}^{\pi} \frac{R d\theta}{\left(\left(\sqrt{y_{d-1}^2+y_d^2}+R\right)^2 + |y|_{d-2}^2 - 4R\sqrt{y_{d-1}^2+y_d^2}\cos^2\left(\frac{\theta}{2}\right)\right)^{\Delta}} \\
&= \frac{2R}{\left[\left(\sqrt{y_{d-1}^2+y_d^2}+R\right)^2 + |y|_{d-2}^2\right]^{\Delta}} \int_{-\pi}^{\pi} \frac{d\theta}{\left(1 - \frac{4R\sqrt{y_{d-1}^2+y_d^2}}{\left(\sqrt{y_{d-1}^2+y_d^2}+R\right)^2+|y|_{d-2}^2}\cos^2\left(\frac{\theta}{2}\right)\right)^{\Delta}} \\
&= \frac{2\pi R}{\left[\left(\sqrt{y_{d-1}^2+y_d^2}+R\right)^2 + |y|_{d-2}^2\right]^{\Delta}} {}_2F_1\left(\frac{1}{2}, \Delta; 1, \frac{4R\sqrt{y_{d-1}^2+y_d^2}}{\left(\sqrt{y_{d-1}^2+y_d^2}+R\right)^2+|y|_{d-2}^2}\right)
\end{aligned}
\tag{C.14}
$$

where at the end we've used $t=\cos^2(\theta/2)$ and the integral representation of the hypergeometric function ${}_2F_1$.

Integral over two angles

$$
\begin{aligned}
\oint_{\mathcal{C}} \frac{dx_1 dx_2}{|x_1-x_2|^{2\Delta}} &= R^{2-2\Delta} \int_{-\pi}^{\pi} \frac{d\theta_2 d\theta_2}{(2(1-\cos(\theta_1-\theta_2)))^{\Delta}} \\
&= 4^{-\Delta} R^{2-2\Delta} 2\pi \int_{-\pi}^{\pi} du \sin^{-2\Delta}\left(\frac{u}{2}\right) \\
&= 4^{-\Delta} R^{2-2\Delta} 8\pi \int_0^{\frac{\pi}{2}} du \sin^{-2\Delta}(u) \\
&= 4^{1-\Delta} R^{2-2\Delta} \pi \frac{\Gamma\left(\frac{1}{2}-\Delta\right)\sqrt{\pi}}{\Gamma(1-\Delta)}
\end{aligned}
\tag{C.15}
$$

C.4 Integration on the sphere

Integration over two angles

$$
\begin{aligned}
\oint_{S_2} \frac{d^2x d^2y}{|x-y|^{2\Delta}} &= \oint d^2y \int_0^{\pi} \int_0^{2\pi} \frac{R^2 \sin\theta d\theta d\phi}{(2R)^{2\Delta} \sin^{2\Delta}\left(\frac{\theta}{2}\right)} \\
&= 2^{3-2\Delta} \pi^2 R^{4-2\Delta} \int_0^{\pi} \frac{\sin\theta d\theta}{\sin^{2\Delta}\left(\frac{\theta}{2}\right)} \\
&= 2^{5-2\Delta} \pi^2 R^{4-2\Delta} \int_0^{\frac{\pi}{2}} \cos\theta \sin^{1-2\Delta} \theta d\theta \\
&= 2^{5-2\Delta} \pi^2 R^{4-2\Delta} \int_0^{1} t^{1-2\Delta} dt \\
&= \frac{(2R)^{4-2\Delta}\pi^2}{1-\Delta}
\end{aligned}
\tag{C.16}
$$

D Difficult diagram in the g-function for the localized magnetic field

$$
\begin{aligned}
D_b &= -\frac{\lambda_0 h_0^4}{4!} N_\sigma^4 \int d^dy \left(\oint \frac{dz}{|y-z|^{2\Delta_\phi}} \right)^4 \\
&= -\frac{\lambda_0 h_0^4}{4!} I
\end{aligned}
\tag{D.1}
$$

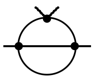
The integral over the circular defect can be conducted using equation (C.14). It simplified to:

$$
I = 2\pi \left(2\pi R\right)^4 N_\sigma^4 S_{2-\varepsilon} \int_0^{\infty} drdz \frac{rz^{3-\varepsilon}}{\left((r+R)^2+z^2\right)^{4-(1+\kappa)\varepsilon}} \left[{}_2F_1\left(\frac{1}{2}, 1-\frac{(1+\kappa)\varepsilon}{4}; 1, \frac{4rR}{(r+R)^2+z^2}\right)\right]^4 \tag{D.2}
$$

It is enough to give I to order $\mathcal{O}(1)$ in ε, since $\lambda_* \sim \varepsilon$ and h_* is finite. The remaining integral can be conducted using appendix B in [2] and it evaluates to $I = 1/(16\pi^2)$.

E Hard diagram in the two-loop renormalization of the bulk

In this section, we adapt the arguments found in [9], which apply to a local scalar field with ϕ^4 interactions, to the RG flow close to the LRI fixed point. The diagram whose poles we're meant to extract is the following:

$$\frac{5N+22}{27} \cdot 3$$ 

$$
\begin{aligned}
&= -\frac{(5N+22)\lambda^3\mu^{3\varepsilon}}{9} \int \frac{d^dp}{(2\pi)^d} \frac{d^dq}{(2\pi)^d} \frac{1}{p^\sigma \left(k_1+k_2-p\right)^\sigma q^\sigma \left(p+k_3-q\right)^\sigma} \\
&= -\frac{(5N+22)\lambda^3\mu^{3\varepsilon}}{9} I
\end{aligned}
\tag{E.1}
$$

Let's focus on the integral at hand – call it I – and forget about numerical constants for the moment. Doing the q integral first using the master integral formulae, we get:

$$
I = \frac{1}{(2\pi)^d} \frac{\left[w_\sigma^{(d)}\right]^2}{w_{2\sigma-d}^{(d)}} \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^\sigma \left(k_1+k_2-p\right)^\sigma \left(p+k_3\right)^{2\sigma-d}} \tag{E.2}
$$

Next, let's introduce Feynman parameters so as to turn the momentum integral into a tadpole integral, whose evaluation is well-known and frequently occuring in QFT (see appendix C.1. As a reminder, Feynman parametrization amounts to using the following property:

$$
\frac{1}{A_1^{\alpha_1}\dots A_n^{\alpha_n}} = \frac{\Gamma\left(\alpha_1+\dots+\alpha_n\right)}{\Gamma(\alpha_1)\dots\Gamma(\alpha_n)} \int_0^1 dx_1 \dots \int_0^1 dx_n \frac{\delta(1-x_1-\dots-x_n) x_1^{\alpha_1-1}\dots x_n^{\alpha_n-1}}{(x_1A_1+\dots x_nA_n)^{\alpha_1+\dots+\alpha_n}} \tag{E.3}
$$

where the A_i can be taken to be scalar functions of momenta. Applying this rule to the three denominators in equation (E.2) yields:

$$
I = \frac{1}{(2\pi)^d} \frac{\left[w_\sigma^{(d)}\right]^2}{w_{2\sigma-d}^{(d)}} \frac{\Gamma\left(2\sigma-\frac{d}{2}\right)}{\Gamma\left(\frac{\sigma}{2}\right)^2 \Gamma\left(\sigma-\frac{d}{2}\right)} \int_0^1 dx_1dx_2dx_3 \int \frac{d^dp}{(2\pi)^d} \frac{\delta\left(1-\sum_i x_i\right)\left(x_1x_2\right)^{\frac{\sigma}{2}-1} x_3^{\sigma-\frac{d}{2}-1}}{\left[x_1p^2+x_2\left(k_1+k_2-p\right)^2+x_3\left(p+k_3\right)^2\right]^{2\sigma-\frac{d}{2}}} \tag{E.4}
$$

The denominator is a power of a quadratic polynomial Q in p. Let's express it in canonical form:

$$\begin{aligned} Q(p) &= (p + x_3 k_3 - x_2 (k_1 + k_2))^2 + x_3 (1 - x_3) k_3^2 + x_2 (1 - x_2) (k_1 + k_2)^2 + 2 x_2 x_3 k_3 (k_1 + k_2) \\ &= \tilde{p}^2 + m^2 \end{aligned} \tag{E.5}$$

where $\tilde{p} = p + x_3 k_3 - x_2 (k_1 + k_2)$ and m^2 denotes the remaining p-independent term. Hence, as promised, the p integral reduces to a tadpole integral of the form:

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 + m^2)^{2\sigma - \frac{d}{2}}} = \frac{(m^2)^{d - 2\sigma}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(2\sigma - d)}{\Gamma\left(2\sigma - \frac{d}{2}\right)} \tag{E.6}$$

Therefore, the integral I reduces to:

$$I = \frac{\left[w_\sigma^{(d)}\right]^2}{w_{2\sigma - d}^{(d)}} \frac{\Gamma(2\sigma - d)}{(2\pi)^d (4\pi)^{\frac{d}{2}} \Gamma\left(\frac{\sigma}{2}\right)^2 \Gamma\left(\sigma - \frac{d}{2}\right)} \int_0^1 dx_i \frac{\delta\left(1 - \sum_i x_i\right) (x_1 x_2)^{\frac{\sigma}{2} - 1} x_3^{\sigma - \frac{d}{2} - 1}}{\left[x_2 (1 - x_2) (k_1 + k_2)^2 + x_3 (1 - x_3) k_3^2 + 2 x_2 x_3 k_3 (k_1 + k_2)\right]^{2\sigma - d}} \tag{E.7}$$

Next, let's conduct the following change of variables:

$$\begin{cases} x_1 = y(1 - z) \\ x_2 = yz \\ x_3 = 1 - y \end{cases} \tag{E.8}$$

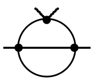
This leads to a new expression of I:

$$I = \frac{\left[w_\sigma^{(d)}\right]^2}{w_{2\sigma - d}^{(d)}} \frac{\Gamma(2\sigma - d)}{(2\pi)^d (4\pi)^{\frac{d}{2}} \Gamma\left(\frac{\sigma}{2}\right)^2 \Gamma\left(\sigma - \frac{d}{2}\right)} \int_0^1 \frac{dy dz y \left[y^2 (1 - z) z\right]^{\frac{\sigma}{2} - 1} (1 - y)^{\sigma - \frac{d}{2} - 1}}{\left[yz (1 - yz) (k_1 + k_2)^2 + y (1 - y) k_3^2 + 2yz (1 - y) k_3 (k_1 + k_2)\right]^{2\sigma - d}} \tag{E.9}$$

Next, using the fact that $\sigma = (d + \varepsilon)/2$ with $\varepsilon > 0$, the poles in this expression can only come from the $y = 1$ part of the above integral.

$$\begin{aligned} I &\supset \frac{\left[w_\sigma^{(d)}\right]^2}{w_{2\sigma - d}^{(d)}} \frac{\Gamma(2\sigma - d)}{(2\pi)^d \Gamma\left(\frac{\sigma}{2}\right)^2 \Gamma\left(\sigma - \frac{d}{2}\right)} \frac{1}{(k_1 + k_2)^{4\sigma - 2d}} \int_0^1 dx_i \left[(1 - z) z\right]^{-\frac{3\sigma}{2} + d - 1} y^{\sigma - 1} (1 - y)^{\sigma - \frac{d}{2} - 1} \\ &= \frac{\left[w_\sigma^{(d)}\right]^2}{w_{2\sigma - d}^{(d)}} \frac{\Gamma(2\sigma - d)}{(2\pi)^d \Gamma\left(\frac{\sigma}{2}\right)^2 \Gamma\left(\sigma - \frac{d}{2}\right)} \frac{\Gamma\left(d - \frac{3\sigma}{2}\right)^2}{\Gamma(2d - 3\sigma)} \frac{\Gamma(\sigma) \Gamma\left(\sigma - \frac{d}{2}\right)}{\Gamma\left(2\sigma - \frac{d}{2}\right)} \frac{1}{(k_1 + k_2)^{4\sigma - 2d}} \\ &= \frac{\left[w_\sigma^{(d)}\right]^2}{w_{2\sigma - d}^{(d)}} \frac{\Gamma(2\sigma - d)}{(2\pi)^d \Gamma\left(\frac{\sigma}{2}\right)^2} \frac{\Gamma\left(d - \frac{3\sigma}{2}\right)^2}{\Gamma(2d - 3\sigma)} \frac{\Gamma(\sigma)}{\Gamma\left(2\sigma - \frac{d}{2}\right)} \frac{1}{k^{4\sigma - 2d}} \end{aligned} \tag{E.10}$$

where in the last line we introduced the total "incoming" momentum $k = k_1 + k_2$. Lastly, reintroducing symmetry factors, we get to the desired result:

$$\frac{5N + 22}{27} \cdot 3$$  $$\supset -\frac{(5N + 22) \lambda^3 \mu^{3\varepsilon}}{9} \frac{\left[w_\sigma^{(d)}\right]^2}{w_{2\sigma - d}^{(d)}} \frac{\Gamma(2\sigma - d)}{(2\pi)^d \Gamma\left(\frac{\sigma}{2}\right)^2} \frac{\Gamma\left(d - \frac{3\sigma}{2}\right)^2}{\Gamma(2d - 3\sigma)} \frac{\Gamma(\sigma)}{\Gamma\left(2\sigma - \frac{d}{2}\right)} \frac{1}{k^{4\sigma - 2d}} \tag{E.11}$$

F Anomalous dimensions from β-function

Let's prove that for an interaction going like $g\mathcal{O}$ in d dimensions, we have:

$$\Delta_{\mathcal{O}} = d + \frac{\partial \beta}{\partial g} \tag{F.1}$$

To prove this, we adapt arguments from [40]. We first show that $T^\mu{}_\mu = \beta_g \mathcal{O}$. Consider an infinitesimal scale transformation $x \mapsto (1 + \varepsilon) x$. In QFT, this also affects the coupling g. Since energy goes like inverse length, we have:

$$g(\mu) \mapsto g\left(\frac{\mu}{1 + \varepsilon}\right) = g(\mu) - \varepsilon \beta_g \tag{F.2}$$

We then require that the action remain unchanged during such a transformation:

$$\delta S = \varepsilon \int d^d x \left[\partial_\mu j^\mu - \frac{\partial \mathcal{L}}{\partial g} \beta_g \right] \tag{F.3}$$

where the first term is the classical variation of the action, expressed using the dilatation current $j^\mu = T^\mu{}_\nu x^\nu$ where $T^{\mu\nu}$ is the stress tensor, and the second is the quantum variation. We've supposed that $\partial \mathcal{L}/\partial g = \mathcal{O}$, hence:

$$\partial_\mu j^\mu - \frac{\partial \mathcal{L}}{\partial g} \beta_g = (\partial_\mu T^\mu{}_\nu) x^\nu + T^\mu{}_\mu - \beta_g \mathcal{O} = 0 \tag{F.4}$$

Next, since one can write $dO/d\log\mu = -\Delta_O O$ for any operator O and using that $\Delta_{T_{\mu\nu}} = d$ is protected, we get on the one hand:

$$\frac{d}{d\log\mu} T^\mu{}_\mu = -d T^\mu{}_\mu = -d \beta_g \mathcal{O} \tag{F.5}$$

and on the other:

$$\frac{d}{d\log\mu} T^\mu{}_\mu = \frac{\partial \beta_g}{\partial g} \beta_g \mathcal{O} - \beta_g \Delta_{\mathcal{O}} \mathcal{O} \tag{F.6}$$

The RHS of the last two equalities being equal to one another leads to the desired result.

G Effective field theory

Let's study the effective theory for ϕ when ψ is integrated out in the final class of theories studied in this work. In this section, we set $\sigma = d - 2\Delta_\phi$ and $\tau = p - 2\Delta_\psi$ It turns out that for $b = 1$ or $b = 2$ – which are the only cases we'll look at – this integration can be carried out exactly. This amounts to the following rewriting of the partition function:

$$Z = \int \mathcal{D}\phi e^{-S_0[\phi]} \int \mathcal{D}\psi e^{-S_1[\phi,\psi]} = \int \mathcal{D}\phi e^{-S_{\text{eff}}[\phi]} \tag{G.1}$$

where $S_0[\phi]$ is the bulk action, $S_1[\phi, \psi]$ is the defect action and the effective action is defined, up to a ϕ-independent term, by:

$$S_{\text{eff}}[\phi] := S_0[\phi] - \log \int \mathcal{D}\psi e^{-S_1[\phi,\psi]} \tag{G.2}$$

G.1 $b = 1$ theory

To integrate out ψ in the $b = 1$ case, one can start by completing the square in the defect action:

$$S_1[\phi, \psi] = \frac{1}{2} \int_p [\psi \mathcal{L}_\tau \psi + g_0 \phi^a \psi] = \frac{1}{2} \int_p \left(\psi + \frac{g_0 \phi^a \mathcal{L}_{-\tau}}{4} \right) \mathcal{L}_\tau \left(\psi + \frac{g_0 \mathcal{L}_{-\tau} \phi^a}{4} \right) - \frac{g_0^2}{16} \int_p \phi^a \mathcal{L}_{-\tau} \phi^a \tag{G.3}$$

A simple translation of ψ makes the first term in S_1 ϕ-independent, and the resulting path integral is a constant. We are hence left with:

$$\begin{aligned} S_{\text{eff}}[\phi] &= \frac{1}{2} \int d^d x \phi \mathcal{L}_\sigma \phi - \frac{g_0^2}{16} \int \phi^a \mathcal{L}_{-\tau} \phi^a \\ &= \frac{1}{2} \int d^d x \phi \mathcal{L}_\sigma \phi - \frac{g_0^2}{16} \mathcal{N}_{2\Delta_\psi - p, p} \int \frac{\phi^a(x) \phi^a(y)}{|x-y|^{p-\tau}} \end{aligned} \tag{G.4}$$

G.2 $b = 2$ theory

In the $b = 2$ case, integrating out ψ amounts to evaluating a Gaussian integral.

$$S_1[\phi] = \frac{1}{2} \int dx \psi \left(\mathcal{L}_\tau + g_0 \phi^a \right) \psi \tag{G.5}$$

This looks like a GFF, $g\phi^a$ generating a mass-like term for ψ. We need to calculate the following functional integral to integrate this field out:

$$\int \mathcal{D}\psi e^{-S_1} = C \det \left(\mathcal{L}_\tau + g_0 \phi^a \right)^{-1/2} = C e^{-\int d^p x L_{\text{eff}}} \tag{G.6}$$

where C is some normalization constant that isn't important and we introduce an effective Lagrangian. Recall [41]:

$$\log\det\hat{A} = \operatorname{Tr}\log\hat{A} \tag{G.7}$$

Hence the difficult task of calculating a determinant can be recast into a trace calculation.

$$\begin{aligned}
\operatorname{Tr}\log\left(\mathcal{L}_\tau + g\phi^a\right) &= \int \frac{d^p k}{(2\pi)^p} \left\langle k\left|\log\left(\mathcal{L}_\tau + g_0\phi^a\right)\right| k\right\rangle \\
&= \int \frac{d^p k}{(2\pi)^p} d^p x e^{ikx} \left\langle x\left|\log\left(\mathcal{L}_\tau + g_0\phi^a\right)\right| k\right\rangle \\
&= cst. + \int \frac{d^p k}{(2\pi)^p} d^p x e^{ikx} \left\langle x\left|\log\left(1 + g_0\mathcal{L}_\tau^{-1}\phi^a\right)\right| k\right\rangle \\
&= cst. + \sum_{j=1}^{+\infty} \frac{g_0^j(-1)^{j+1}}{j} \int d^p x \frac{d^p k}{(2\pi)^p} e^{ikx} \left\langle x\left|\left[\mathcal{L}_\tau^{-1}\phi^a\right]^j\right| k\right\rangle
\end{aligned} \tag{G.8}$$

One can then use the following identity, easily derived using the action of the fractional Laplacian on plane waves.

$$\left\langle x\left|\left[\mathcal{L}_\tau^{-1}\phi^a\right]^j\right| k\right\rangle = e^{-ikx} \int \prod_{i=1}^{j} \frac{d^p q_i}{(2\pi)^p} \hat{\phi^a}(q_i) e^{iq_i x} \frac{1}{|q_1 + \cdots + q_i - k|^\tau} \tag{G.9}$$

We finally get $L_{\text{eff}} = \sum_j L_{\text{eff}}^{(j)}$ with

$$\begin{aligned}
L_{\text{eff}}^{(j)} &= \frac{g_0^j(-1)^{j+1}}{2j} \int \frac{d^p k}{(2\pi)^p} \prod_{i=1}^{j} \frac{d^p q_i}{(2\pi)^p} \hat{\phi^a}(q_i) e^{iq_i x} \frac{1}{|q_1 + \cdots + q_i - k|^\tau} \\
&= \frac{g_0^j(-1)^{j+1}}{2j} \phi(x)^a \int \frac{d^p k}{(2\pi)^p} \frac{1}{|k|^\tau} \prod_{i=2}^{j} \frac{d^p q_i}{(2\pi)^p} \hat{\phi^a}(q_i) e^{iq_i x} \frac{1}{|q_2 + \cdots + q_i - k|^\tau} \\
&= \frac{g_0^j(-1)^{j+1}}{2j} \phi(x)^a \int \frac{d^p k}{(2\pi)^p} \frac{1}{|k|^\tau} \prod_{i=2}^{j} \frac{d^p q_i}{(2\pi)^p} d^p x_i \phi(x_i)^a e^{iq_i(x-x_i)} \frac{1}{|q_2 + \cdots + q_i - k|^\tau}
\end{aligned} \tag{G.10}$$

Reorganizing the complex exponentials allows one to rewrite this using known propagators, yielding:

$$S_{\text{eff}} = \sum_{j=1}^{+\infty} \frac{(-1)^{j+1} N_\psi^j}{2j} g_0^j \int \prod_{i=1}^{j} d^p x_i \frac{\phi(x_i)^a}{|x_{i+1} - x_i|^{p-\tau}} \tag{G.11}$$